

Proof of Pak's Conjecture on Tilings by T-Tetrominoes

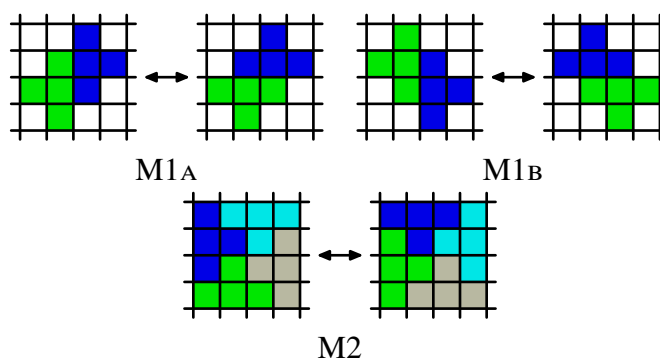
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Abstract

We prove Igor Pak's conjecture on tilings of simply connected regions by T-tetrominoes in the case where the region is a rectangle. The problem was solved independently by Michael Korn and Igor Pak [KP].

Pak's Conjecture [Pak]. *Any tiling of a simply connected region by T-tetrominoes can be transformed into any other one by a sequence of local moves of the following form:*



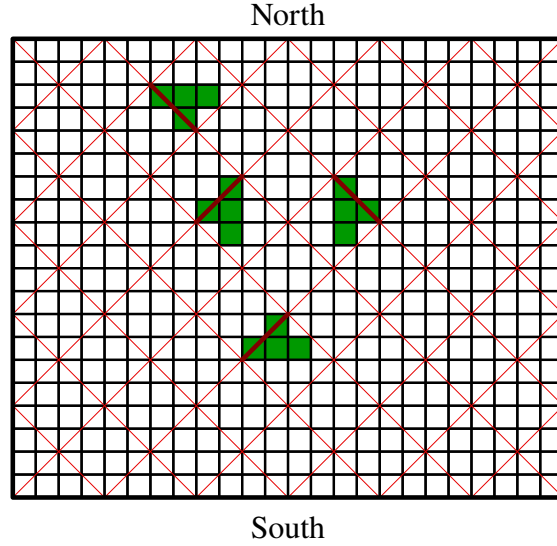
The proof consists of several parts. First we study arbitrary tilings of a rectangle by tetrominoes; then we reduce the tiling problem (for a rectangle) to a dual problem (the six-vertex model). Finally, using the theorem of [Eloranta] on local moves in the six-vertex model, we obtain the desired result.

For convenience, we use the following names for the directions: *north* (up), *south* (down), *west* (left), *east* (right).

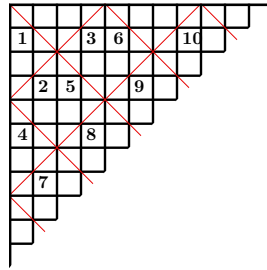
Consider an arbitrary $4n \times 4m$ rectangle and an arbitrary tiling of it. Draw diagonals in it with a step of 4 cells (see the figure). This divides the rectangle into squares (tilted by 45°). For brevity, we call the sides of these squares edges. We say that an edge is regular (in a given tiling) if it crosses exactly one tetromino. The figure below shows 4 regular edges (the grid of edges is drawn in red, the regular edges themselves in dark red, and the tetrominoes crossing them in green). We call a tiling regular if all edges in this tiling are regular. We want to prove that all tilings are regular.

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We number the edges starting from the north-western (upper left) corner and moving from south-west to north-east:



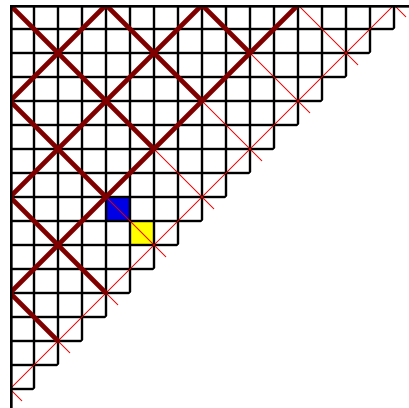
Lemma 1. *Let P be a tiling in which the first n edges are regular. Then the $(n + 1)$ -st edge is regular as well.*

Proof. Consider the $(n + 1)$ -st edge in the tiling P and let us show that it is regular. Suppose to the contrary that the $(n + 1)$ -st edge is not regular, i.e., that two different tetrominoes pass through the two cells of the edge.

This edge can be either “vertical” – running from north-west to south-east – or “horizontal” – running from south-west to north-east. We consider these two cases separately.

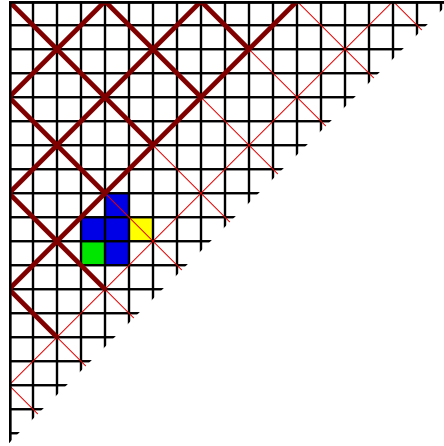
Vertical edge

In the figures below, the first n edges are marked by bold dark red lines (we know that these edges are regular), and the $(n + 1)$ -st edge is colored blue and yellow.



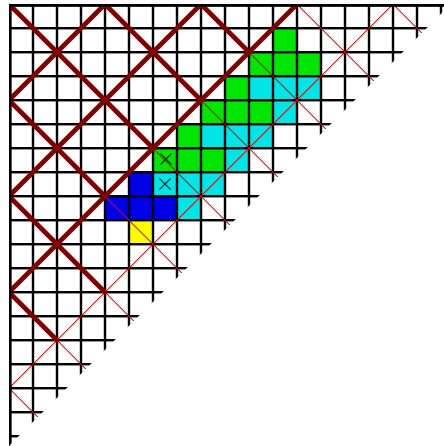
Consider the tetromino passing through the blue cell (see the figure). It does not cross the regular edges and does not pass through the yellow cell (the second cell of the edge). Hence, a priori, the following cases are possible:

CASE V1:



In this case no tetromino can contain the green cell (it is essential that to the south-west of this cell there is a regular edge, whose cells belong to one and the same tetromino).

CASE V2:



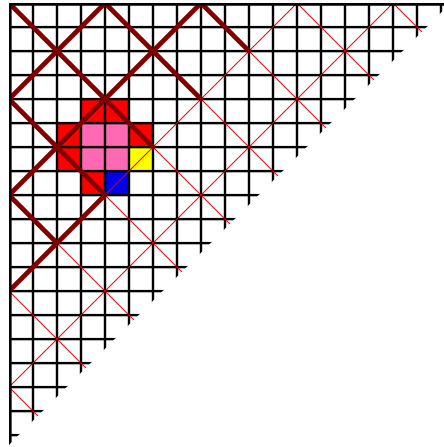
If the tetromino is positioned as in the figure above, let us see which tetrominoes pass through the cells marked with crosses. The only possibility (taking into account that a tetromino can cross the regular (bold) edges only in two cells) is the tetrominoes drawn in light blue and green. But the green tetromino is positioned exactly like the blue one, so, considering the tetrominoes to the north-east, we obtain a sequence of light blue and green tetrominoes. However, this sequence must cross the vertical or the horizontal wall of the rectangle (the eastern or the northern wall, respectively). Hence such a tiling is impossible.

Horizontal edge

Fact. Every tetromino crossing the pink square (see the figure) crosses it in exactly two cells.

Indeed, if this tetromino crosses one of the three regular edges of the pink square, then two of its cells lie on this regular edge, and the other two must lie on the same side of the edge, i.e., both lie in the pink square. Since every cell of the pink square is covered by some tetromino, and the tetrominoes passing through the regular edges cover an even number (0 or 2) of cells of the square,

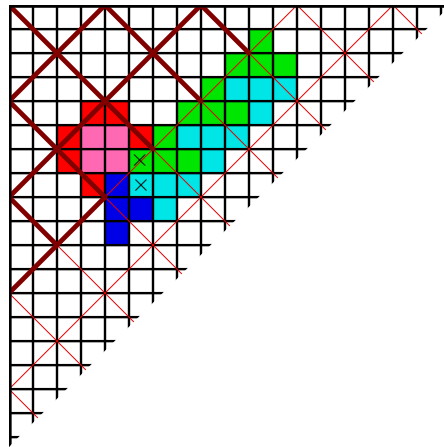
the tetromino passing through the fourth edge must cover an even number of cells as well (although two tetrominoes do pass through the fourth (non-regular) edge, it is clear that only one of them can cross the pink square).



As in the case of a vertical edge, consider the tetromino passing through the blue cell. Observe that if it crosses the pink square in two cells, then it also contains the yellow cell, and hence this edge is regular. Therefore this tetromino cannot contain any cell of the pink square.

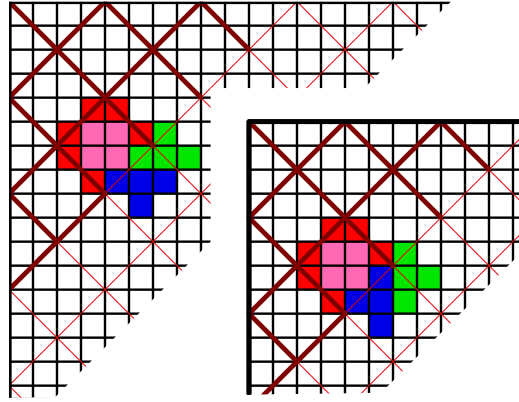
Let us consider the remaining positions of this tetromino (there are two of them):

CASE H1:



If the blue tetromino is positioned as in the figure, we obtain a case analogous to V2. Considering the tetrominoes passing through the cells marked with crosses, we see that the continuation of the tiling to the north-east is uniquely determined (as in the figure). However, such a continuation must cross the northern or the eastern boundary.

CASE H2:



If the tetromino is positioned as in this figure, we perform the replacement (move) shown here and arrive at case H1. (That is, we modify our tiling, and then case H1 shows that it is impossible.)

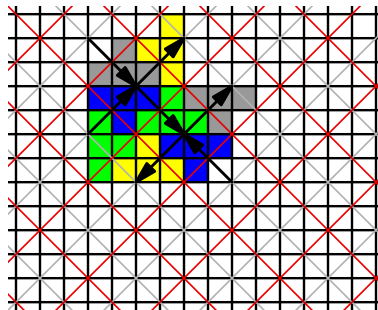
Thus we have shown that the $(n + 1)$ -st edge is regular. $\square$

Applying Lemma 1 and using induction, we obtain:

Theorem 1. *Every tiling of a $4n \times 4m$ rectangle by tetrominoes is regular.*

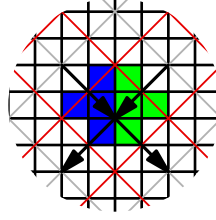
Remark. This theorem easily implies the theorem of [Walkup] that tetrominoes can tile only rectangles of the form $4n \times 4m$. Indeed, suppose that a $k \times l$ rectangle (k or l not divisible by 4) can be tiled by tetrominoes. Repeating this tiling several times, we can tile a $4k \times 4l$ rectangle. Theorem 1 already applies to this tiling; that is, this tiling is regular. Now, if we consider the original tiling, it must be regular as well, but this is impossible because of a “boundary” effect: in the north-eastern corner, an edge cuts off a region that cannot be tiled.

Now let us see how the squares bounded by the edges are covered. As we already know, each edge of a square corresponds to exactly one tetromino. This tetromino may lie in either of the two adjacent squares. We say that it is directed from the square that does not contain it to the square that does. Thus every tiling corresponds to an assignment of arrows to the edges of the “dual” lattice (drawn in gray in the figure), whose vertices are the centers of the squares. Moreover, exactly two arrows point into each vertex.

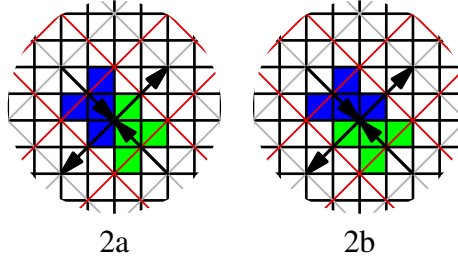


Conversely, it is easy to see that every assignment of arrows (in which exactly two arrows point into each vertex of the dual lattice) corresponds to some tiling by tetrominoes. It suffices to verify this locally: we need to show that every combination of arrows pointing into the center of a square corresponds to some tiling of the square itself. Moreover, up to rotation there are only two different configurations of incoming arrows:

1. Two adjacent arrows point inward:



2. Two diametrically opposite arrows point inward. This configuration corresponds to two tilings (which are transformed into each other by the moves M1A and M1B):

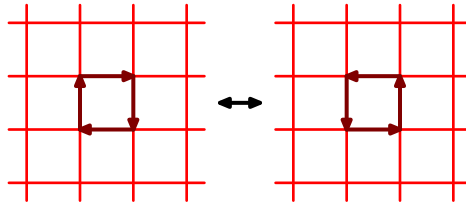


To make the problem of assigning arrows equivalent to the problem of tiling by tetrominoes, we identify tilings that differ only by the moves M1A and M1B. Thus we have reduced the tiling problem to the following problem:

The “Six-vertex model” problem. Given a lattice, assign arrows to its edges so that exactly two arrows enter each vertex.

Let us prove the following theorem.

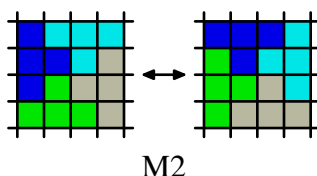
Theorem 2 [Eloranta]. *Let a lattice in a simply connected region be given. Then any assignment of arrows can be transformed into any other one by a sequence of local moves that reverse the arrows of a cycle around a single cell.*



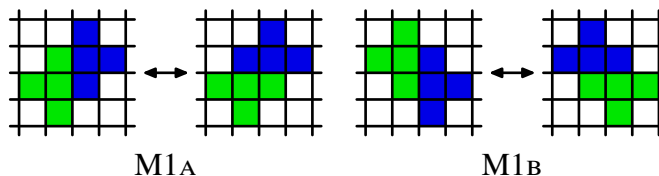
Proof. We introduce a height function $h(x)$ on the cells of the lattice. As we move from one cell to another, h increases by 1 if we cross an arrow running from left to right (relative to our direction of motion), and decreases by 1 otherwise. This function is well defined, since if we traverse a closed cycle, the number of arrows leaving the cycle equals the number of arrows entering it (this follows from the fact that exactly two arrows enter each vertex and exactly two leave it, so a conservation law holds: as many arrows “flow into” each cycle as “flow out” of it). On the boundary we set $h(x)$ equal to 0.

Consider an arbitrary assignment of arrows. Using local moves, we bring it to one of the local minima (one assignment of arrows is less than or equal to another if the height function of the first is less than or equal to the height function of the second at every cell). We show that the height function of the resulting assignment has no local maxima. Indeed, if a cell s were a local maximum, then a cycle would have to go around it (since $h(x)$ decreases in all directions); reversing the direction of this cycle, we would decrease the value $h(s)$ by 2 and leave the values at all other cells unchanged. So, $h(x)$ equals 0 on the boundary and has no local maxima inside the region; hence $h(x)$ is strictly negative inside. It follows that all arrows on the boundary are directed clockwise. Similarly, the arrows inside also form concentric cycles. Thus we have shown that all assignments of arrows that are local minima are the same. $\square$

Let us now pass from the dual model back to the model of tilings by tetrominoes. It is easy to see that the reversal of a cycle corresponds to the following local move:



Thus this move, together with the two moves by which we identified tilings:



forms a system of local moves for tilings of a rectangle by tetrominoes. This proves Pak's conjecture in the special case of rectangles.

References

- [Eloranta] Kari Eloranta, Diamond Ice, Jour. of Stat. Phys., 1999
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