Problem Set 2
Computational and Metric Geometry

You can discuss homework problems with other students but you must write solutions on your own. This homework is due on Monday, February 25. It is sufficient to solve problems 1, 2.1, 2.2, and 3.1 to get the full score.

Problem 1. Let A be a linear operator from ℓ_2^d to ℓ_2^d. Suppose that A has singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d > 0$. Compute the distortion of A.

Problem 2.

1. Prove that there exists an embedding of ℓ_1^d in ℓ_2^d with distortion $\sqrt{d}$.
2. Suppose that f is a linear map from ℓ_1^d to ℓ_2^d. Prove that f has distortion at least $\sqrt{d}$.

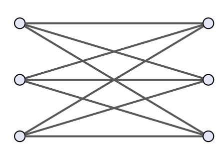
   Hint: Consider the standard basis $e_1, \dots, e_d$ of ℓ_1^d. Let $r_1, \dots, r_d \in \{\pm 1\}$ be independent unbiased Bernoulli random variables. Choose an index $j \in \{1, \dots, d\}$ uniformly at random. Define r'_i by $r'_i = r_i$ if $i \neq j$, and $r'_i = -r_i$ if $i = j$. (That is, $r'_i = r_i$ for all but one index i.) Define random variables u and u' as follows: $u = \sum_{i=1}^d r_i e_i$ and $u' = \sum_{i=1}^d r'_i e_i$. Compute the values of
   $$\frac{\mathbb{E}\left[\|u - (-u)\|_1^2\right]}{\mathbb{E}\left[\|u - u'\|_1^2\right]} \quad \text{and} \quad \frac{\mathbb{E}\left[\|f(u) - f(-u)\|_2^2\right]}{\mathbb{E}\left[\|f(u) - f(u')\|_2^2\right]}.$$

3*. (extra credit) Suppose that f is a differentiable bijective map from ℓ_1^d to ℓ_2^d (f is not necessarily linear). Prove that f has distortion at least $\sqrt{d}$.

4**. (extra credit) By Rademacher's theorem, every Lipschitz map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is differentiable almost everywhere. Using Rademacher's theorem, prove that every map from ℓ_1^d to ℓ_2^d has distortion at least $\sqrt{d}$.

Problem 3.

1. Recall that $K_{3,3}$ is the complete bipartite graph with parts of size 3.



Let G be the graph obtained from $K_{3,3}$ by replacing every edge with a path of length n. Show that every embedding of the shortest path metric on G into the Euclidean plane, has distortion $\Omega(n)$.

2*. (extra credit) Prove that every metric space on n points embeds into $\mathbb{R}$ with distortion $O(n)$.

Problem 4^* **[extra credit].** Consider two metric spaces (X, d_X) and (Y, d_Y). Let $A \subset X$ be a subset of X and $f : A \to Y$ be a Lipschitz map from A to Y. We say that a map $\tilde{f} : X \to Y$ is an extension of f if $\tilde{f}(x) = f(x)$ for every $x \in A$. The Lipschitz extendability constant $e_k(X, Y)$ is the infimum over all numbers C such that the following property holds: for every $A \subset X$ of size at most k and every map $f : A \to Y$ there exists an extension $\tilde{f} : X \to Y$ of f with $\|\tilde{f}\|_{Lip} \leq C\|f\|_{Lip}$.

1. Give an example of two normed spaces $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ such that $e_3(U, V) > 1$.
2. Prove that for every metric space (X, d_X) and every k, $e_k(X, \mathbb{R}) = 1$.
3. Prove that for every metric space (X, d_X), k and N, $e_k(X, \ell_\infty^N) = 1$.

You may assume that X is a finite metric space.