Problem Set 2
Computational and Metric Geometry

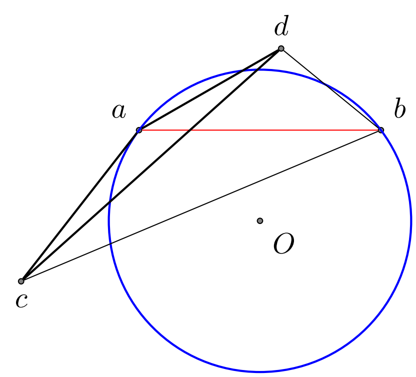
February 7, 2017

You can discuss problems with other students, but you must write solutions on your own. This homework is due on Wednesday, February 22.

Problem 1. Consider a set of points P. Assume that not all of them lie on the same line, but do not assume that they are in general position. Recall that the Delaunay graph $G_D = (P, E_D)$ for P is the dual graph of the Voronoi diagram of P. That is, $(a, b) \in E_D$ if and only if Voronoi cells $V(a)$ and $V(b)$ are adjacent.

1. Prove that every Delaunay triangulation contains all the edges from E_D.

   Hint: Assume that this is not the case. Then there is a Delaunay triangulation $\mathcal{T}$ and edge $(a, b) \in E_D$, such that (a, b) is not an edge in $\mathcal{T}$. Argue that there should be an edge (c, d) in $\mathcal{T}$ that intersects (a, b); further, we may assume that $\triangle acd$ is in $\mathcal{T}$. Since $(a, b) \in E_D$, there is a circle C passing through a and b such that every point of P other than a and b lies strictly outside of C. In particular, c and d lie outside of C. Show that the following facts are inconsistent: (1) segments (a, b) and (c, d) intersect; (2) a and b lie on C, but c and d lie outside of C; (3) $\angle cad + \angle cbd \leq \pi$ (How do we know that $\angle cad + \angle cbd \leq \pi$?).



2. Consider a face of the Delaunay graph. It is a polygon. Prove that all vertices of the polygon lie on a circle.

3. Conclude that any triangulation that contains all the edges of the Delaunay graph is a Delaunay triangulation.

Problem 2. Let P be a set of points in the plane. Recall that all triangulations for P have the same number of triangles. Denote their number by m. For each triangulation $\mathcal{T}$, define a tuple $\alpha_{\mathcal{T}}$ of $3m$ numbers as follows: the entries of $\alpha_{\mathcal{T}}$ are the angles of all triangles in $\mathcal{T}$ sorted from the smallest to the largest. Consider the lexicographic order on the set of tuples $\alpha_{\mathcal{T}}$. Let us say that a triangulation T for P is an angle optimal triangulation if $\alpha_{\mathcal{T}} \geq \alpha_{\mathcal{T}'}$ for every triangulation $\mathcal{T}'$ for P.

1. Prove that every angle optimal triangulation is a Delaunay triangulation. Hint: assume that the Dalaunay property is violated for some pair of triangles $\triangle abc$ and $\triangle abd$; consider the triangulation $\mathcal{T}'$ that we obtain from $\mathcal{T}$ by replacing $\triangle abc$ and $\triangle abd$ with $\triangle cda$ and $\triangle cdb$.

2. Give an example of a set P for which there is more than one angle optimal triangulation.

3. Give an example of a Delaunay triangulation that is not angle optimal.

Problem 3. Let P be a set of points in the plane, and $\mathcal{D}$ be the Voronoi diagram of P. Suppose that $\mathcal{D}$ has k vertices, and their degrees are $d_1, \ldots, d_k$. How many Delaunay triangulations for P are there? (Hint: use Catalan numbers in your solution.)