Homework Assignment 2

TTIC 31010/CMSC 37000-1

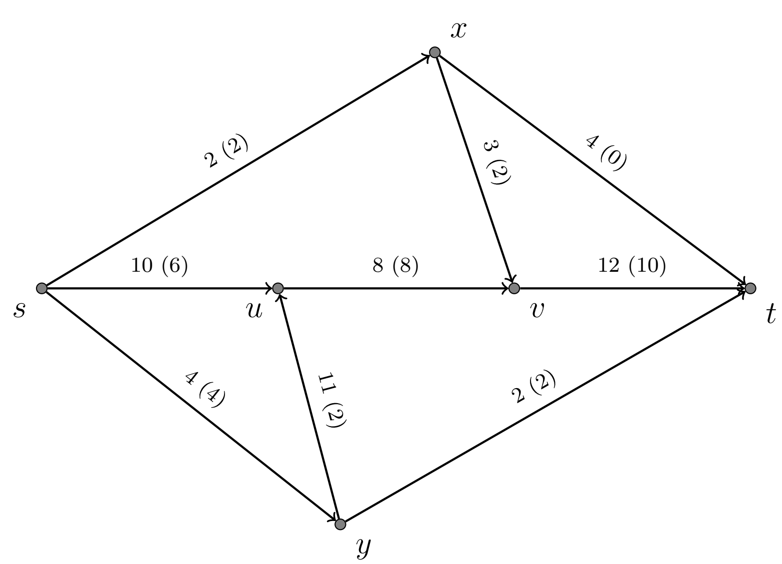
Submit the assignment at the beginning of the class on February 16. You can discuss the problems with other students taking the class. However, you must write your solutions yourself. Please, do not look up solutions online.

Problem 1. Let $G = (V, E)$ be a flow network with a source s, sink t, and positive integer edge capacities $c(e)$. Decide whether each of the following statements is true or false. If a statement is true, give a proof; if it is false, provide a counterexample.

- If all capacities $c(e)$ are even, then the value of the maximum flow is even.
- If all capacities $c(e)$ are odd, then the value of the maximum flow is odd.
- If f is a maximum s-t flow in G, then f saturates every edge in $out(s)$ with flow. That is, for each $e \in out(s)$, $f(e) = c(e)$.

Problem 2. Consider the flow network G shown in the figure below. For every edge e, its capacity $c(e)$ and its flow value $f(e)$ are written next to the edge ($f(e)$ appears in parentheses).

- Draw the residual graph G_f (draw all forward and backward edges of G_f).
- Write the residual capacity of every edge e of G_f (you can write it next to the drawing of the edge).



Problem 3. Consider a flow network $G = (V, E)$ with positive edge capacities $\{c(e)\}$. Let $f : E \to \mathbb{R}_{\geq 0}$ be a maximum flow in G, and G_f be the residual graph. Denote by S the set of nodes reachable from s in G_f and by T the set of nodes from which t is reachable in G_f. That is,

$$S = \{u : \text{ there is a directed path from } s \text{ to } u \text{ in } G_f\},$$
$$T = \{v : \text{ there is a directed path from } v \text{ to } t \text{ in } G_f\}.$$

Prove that $V = S \cup T$ if an only if G has a *unique* s-t minimum cut (an s-t cut whose capacity is strictly less than the capacity of any other s-t cut).

Problem 4. Consider a bipartite graph $G = (X \cup Y, E)$ with parts X and Y. Each part contains $2k$ vertices (i.e. $|X| = |Y| = 2k$). Suppose that $\deg u \geq k$ for every $u \in X \cup Y$. Prove that G has a perfect matching.