Homework Assignment 4

TTIC 31010

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Problem 1. Give an example of a linear program such that both the primal and dual are infeasible.

Problem 2. Convert the following LP to the canonical form. Then write the dual LP.

maximize: $x + 2y - w$

subject to

$$x - y \geq 5$$
$$x + y + z \leq 3$$
$$w \leq 7$$
$$x \geq 0$$
$$y \geq 0$$
$$w \geq 0.$$

Is this LP feasible? If it is feasible, is it bounded or unbounded? Is the dual feasible? If it is feasible, is it bounded or unbounded?

Problem 3. We will design an algorithm that finds a minimum cut in a directed graph using linear programming (LP). Suppose that we are given a directed graph $G = (V, E)$, source s, sink t and a set of edge capacities $\{c(e)\}$. In class, we showed that the dual to the Maximum Flow linear program is a linear program for the Minimum Cut Problem. Specifically, we derived the following linear program for the Minimum Cut Problem:

minimize: $\sum_{e \in E} c(e) y_e$

subject to

$$d_u - d_v + y_{(u,v)} \geq 0 \quad \text{for every edge } (u, v) \in E$$
$$d_s = 0$$
$$d_t = 1$$
$$y_e \geq 0 \quad \text{for every edge } e \in E$$

Then we showed that for every s-t cut (A, B) there is a corresponding feasible solution (d, y) whose value equals the capacity of the cut (A, B). We concluded, using LP duality, that the solution that corresponds to a minimum cut is an optimal solution for this linear program. In this problem, we will develop an algorithm that given any feasible solution (d, y) finds a cut whose capacity is less than or equal to the LP value $\sum_{e\in E} c(e)y_e$.

Consider an arbitrary feasible solution (d, y). Let $p \in [0, 1)$. Define $A_p = \{u : d_u \leq p\}$ and $B_p = \{u : d_u > p\}$.

1. Prove that (A_p, B_p) is an s-t cut.
2. Consider the following function (where $(u, v) \in E$ is an edge)

$$I_{(u,v)}(x) = \begin{cases} 1, & \text{if } d_u \leq x < d_v \\ 0, & \text{otherwise} \end{cases}$$

   Prove that the capacity of the cut (A_p, B_p) equals $\sum_{e\in E} c(e)I_e(p)$.

3. Prove that for every edge $e \in E$, we have $\int_0^1 I_e(x)dx \leq y_e$.
4. Let $g(x) = \text{cap}(A_x, B_x)$. Prove that $\int_0^1 g(x)dx \leq \sum_{e\in E} c(e)y_e$.
5. Conclude that for some $x \in (0, 1)$, $\text{cap}(A_x, B_x) = g(x) \leq \sum_{e\in E} c(e)y_e$.
6. Prove that $g(x)$ is a piece-wise constant function. Find all discontinuities of $g(x)$. Design an efficient algorithm that finds x at which $g(x)$ attains its minimum in $[0, 1)$.
7. Design an efficient algorithm that given an optimal solution to the linear program for the Minimum Cut Problem, returns a minimum cut.

Definition 1. Suppose that we are given an underected graph $G = (V, E)$. A *vertex cover* is a subset of vertices C such that every edge of G is incident to at least one vertex in C. A minimum vertex cover is a vertex cover of smallest possible size. The Minimum Vertex Cover Problem asks to find a minimum vertex cover.

Problem 4. In this problem, we will design an algorithm for solving the Minimum Vertex Cover Problem in bipartite graphs. Let $G = (X \cup Y, E)$ be a bipartite graph with parts X and Y. Consider the following linear programming formulation of the Minimum Vertex Cover Problem. There is an LP variable x_u for every vertex $u \in X \cup Y$.

$$\text{minimize:} \quad \sum_{u\in X\cup Y} x_u$$

subject to

$$x_u + x_v \geq 1 \quad \text{for every edge } (u, v) \in E$$
$$x_u \geq 0$$

1. Prove that the value of this linear program is at most the minimum vertex cover size.
2. Let $\hat{x}$ be an optimal solution. Prove that $\hat{x}_u \in [0, 1]$ for every $u \in X \cup Y$.
3. Assume additionally that $\hat{x}$ is a vertex of the feasible polytope. Prove that then $\hat{x}_u \in \{0, 1\}$. To this end, consider the set of vertices A:

$$A = \{u \in X \cup Y : x_u \neq 0 \text{ and } x_u \neq 1\}.$$

   We need to prove that A is empty. Assume to the contrary that A is not empty. Denote $\varepsilon = \min \{\hat{x}_u, 1 - \hat{x}_u : u \in A\}$. Consider solutions x' and x'' defined by

$$x'_u = \begin{cases} \hat{x}_u, & \text{if } u \notin A \\ \hat{x}_u + \varepsilon, & \text{if } u \in A \cap X \\ \hat{x}_u - \varepsilon, & \text{if } u \in A \cap Y \end{cases} \qquad x''_u = \begin{cases} \hat{x}_u, & \text{if } u \notin A \\ \hat{x}_u - \varepsilon, & \text{if } u \in A \cap X \\ \hat{x}_u + \varepsilon, & \text{if } u \in A \cap Y \end{cases}$$

   Prove that x' and x'' are feasible solutions. Conclude that x is not a vertex of the feasible polytope.
4. Design an algorithm that solves the Minimum Vertex Cover problem in bipartite graphs. The algorithm may use a subroutine that finds an optimal solution $\hat{x}$ that is a vertex of the feasible polytope.