Homework Assignment 3

TTIC 31010

February 13, 2014

Problem 1. Let G be a flow network, $\{c(e)\}$ be a set of edge capacities, and $f : E \to \mathbb{R}_{\geq 0}$ be a maximum flow in the network G with capacities $c(e)$. Consider another set of capacities $\{c'(e)\}$ s.t.

- $c'(e) = c(e)$ for every edge e saturated by f (i.e. for every $e \in E$ s.t. $f(e) = c(e)$),
- $c'(e) \geq c(e)$ for every edge $e \in E$.

Prove that f is a maximum flow in G with capacities $\{c'(e)\}$.

Problem 2. Consider a bipartite graph $G = (X \cup Y, E)$. Suppose that G is a k regular graph; that is, the degree of every vertex equals k.

- Prove that G has a perfect matching.
- Prove that, moreover, there are k disjoint perfect matchings in G. That is, there are perfect matchings $M_1, \ldots, M_k$ s.t. $M_i \cap M_j = \varnothing$ for every $i \neq j$.

Problem 3. Consider the following job assignment problem. There are n people and N jobs. We are given sets $S_1, \ldots, S_n$ and numbers $k_1, \ldots, k_n$. Our goal is to assign jobs to people so as to maximize the number of assigned jobs subject to the following conditions:

- All jobs assigned to person i are in S_i (for every i): $\{j : \text{ job } j \text{ is assigned to } i\} \subseteq S_i$.
- At most k_i jobs are assigned to person i (for every i): $|\{j : \text{ job } j \text{ is assigned to } i\}| \leq k_i$.

Design a polynomial-time that solves this problem.

Problem 4. In this question we will help a hospital figure out whether it has enough supplies for blood transfusions for its patients. There are x_A patients with blood type A, x_B patients with blood type B, x_{AB} patients with blood type AB, and x_O patients with blood type O currently at the hospital, and each patient needs a transfusion of one unit of blood. The hospital has at its disposal s_A units of blood of type A, s_B of type B, s_{AB} of type AB and s_O of type O. The rules of blood transfusion are as follows:

- Patients with blood type A can receive only blood of types A or O.

- Patients with blood type B can receive only blood types B or O.
- Patients with blood type O can receive only blood of type O.
- Patients with blood type AB can receive any of the four types.

Design an efficient algorithm that determines whether the hospital's blood supply is sufficient for treating the patients, and if so, computes a way to distribute the hospital supplies among the patients, so that each of them receives blood of an appropriate type.