# Lecture 16
# Deep Neural Generative Models

## CMSC 35246: Deep Learning

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- We don't construct a probabilistic model of the data
- Can't sample from the model

# Representations

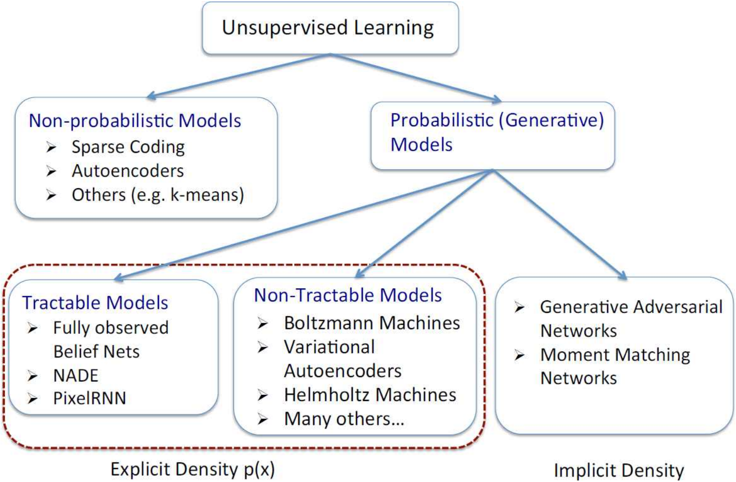


Figure: Ruslan Salakhutdinov

- To motivate Deep Neural Generative models, like before, let's seek inspiration from simple linear models first

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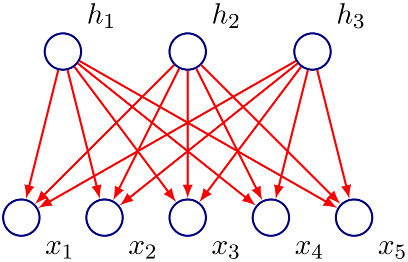
- Formally: Suppose we sample the latent factors from a distribution $\mathbf{h} \sim P(\mathbf{h})$
- Then: $\mathbf{x} = W\mathbf{h} + \mathbf{b} + \epsilon$

# Linear Factor Model

- $P(\mathbf{h})$ is a factorial distribution




$$\mathbf{x} = W\mathbf{h} + \mathbf{b} + \epsilon$$

- How do learn in such a model?
- Let's look at a simple example

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- Suppose underlying latent factor has a Gaussian distribution

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- We care about the marginal $P(\mathbf{x})$ (predictive distribution):

$$P(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\mathbf{b}, WW^T + \sigma^2 I)$$

## Probabilistic PCA

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- Let's look at the ML Estimation:
- Let $C = WW^T + \sigma^2 I$
- We want to maximize $\ell(\theta; X) = \sum_i \log P(\mathbf{x}_i|\theta)$

# Probabilistic PCA: ML Estimation

$$\ell(\theta; X) = \sum_i \log P(\mathbf{x}_i|\theta)$$

$$= -\frac{N}{2}\log|C| - \frac{1}{2}\sum_i (\mathbf{x}_i - \mathbf{b})C^{-1}(\mathbf{x}_i - \mathbf{b})^T$$

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- Now fit the parameters $\theta = W, \mathbf{b}, \sigma$ to maximize log-likelihood
- Can also use EM

# Factor Analysis

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- Still consider linear relationship between inputs and observed variables: Marginal $P(\mathbf{x}) \sim \mathcal{N}(\mathbf{x}; b, WW^T + \Psi)$

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- Parameters are coupled, makes ML estimation difficult
- Need to employ EM (or non-linear optimization)

# More General Models

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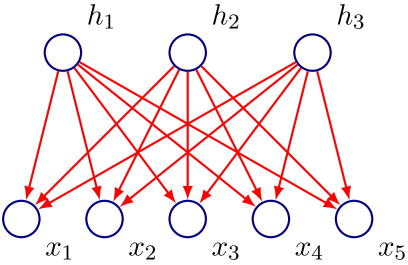
- The decoding of the input from the latent states can be a complicated non-linear function
- Estimation and inference can get complicated!

# Earlier we had:



# Quick Review

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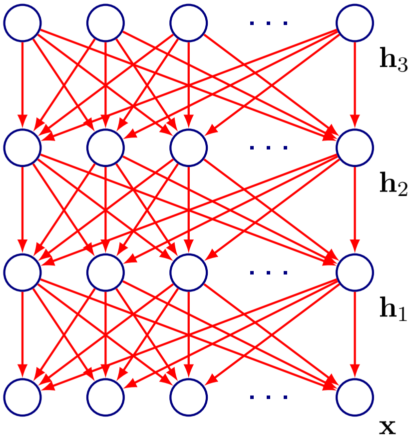
- The nodes represent random variables and arcs indicate dependency
- Some of the random variables are observed, others are hidden

# Sigmoid Belief Networks




- Just like a feedfoward network, but with arrows reversed.

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- Marginalization yields $P(\mathbf{x})$, intractable in practice except for very small models

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- Deep Belief Networks are like Sigmoid Belief Networks except for the top two layers

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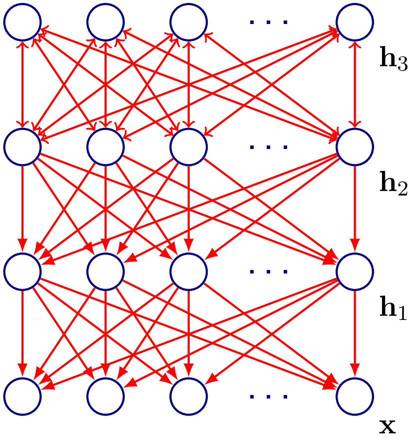
  - R. M. Neal: Connectionist Learning of Belief Networks, In **Artificial Intelligence**, 1992

# Deep Belief Networks




- The top two layers now have undirected edges

# Deep Belief Networks

- The joint probability changes as:

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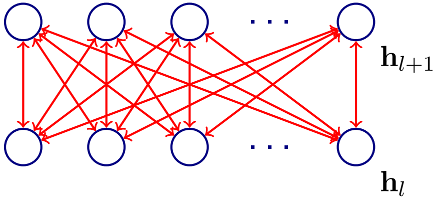


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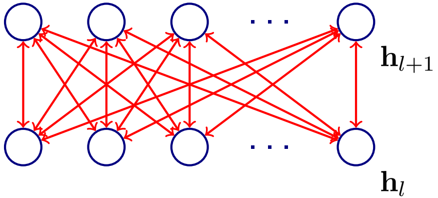


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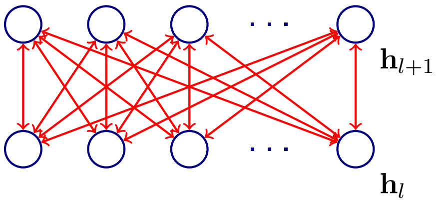


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$$P(\mathbf{h}^{l+1}, \mathbf{h}^{l}) \propto \exp(\mathbf{b}'\mathbf{h}^{l-1} + \mathbf{c}'\mathbf{h}^{l} + \mathbf{h}^{l} W \mathbf{h}^{l-1})$$

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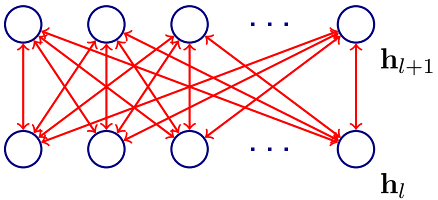


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- We will return to RBMs and training procedures in a while, but first we look at the mathematical machinery that will make our task easier

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- Energies are in the log-probability domain:

$$\text{Energy}(\mathbf{x}) = \log \frac{1}{(ZP(\mathbf{x}))}$$

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- How do we specify the energy function?

# Product of Experts Formulation

- In this formulation, the energy function is:

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- We have the product of experts:

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- Contrast this with mixture models

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- We introduce another term in analogy from statistical physics: free energy:

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- Free Energy is just a marginalization of energies in the log-domain:

$$\text{FreeEnergy}(\mathbf{x}) = -\log \sum_{\mathbf{h}} \exp^{-(\text{Energy}(\mathbf{x},\mathbf{h}))}$$

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- We have an expression for $P(\mathbf{x})$ (and hence for the data log-likelihood). Let us see how the gradient looks like

# Data Log-Likelihood Gradient

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# Data Log-Likelihood Gradient

- The expected log-likelihood gradient over the training set has the following form:

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- Resort to Markov Chain Monte Carlo to get a stochastic estimator of the gradient

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- Suppose the energy has the following form:

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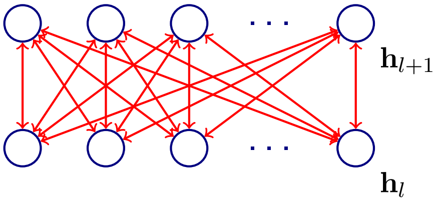
$$\begin{aligned} \text{FreeEnergy}(\mathbf{x}) &= -\log P(\mathbf{x}) - \log Z \\ &= -\beta - \sum_i \log \sum_{\mathbf{h}_i} \exp^{-\gamma_i(\mathbf{x},\mathbf{h}_i)} \end{aligned}$$

# Restricted Boltzmann Machines

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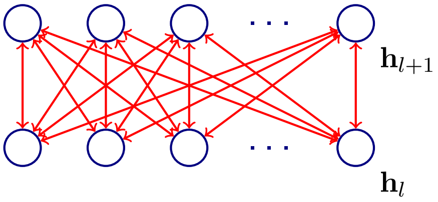


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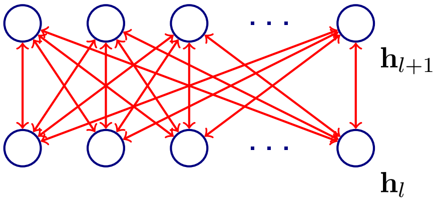


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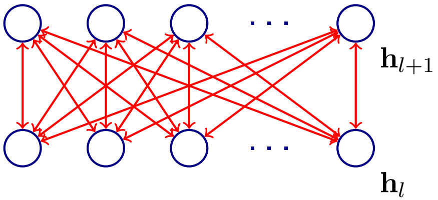


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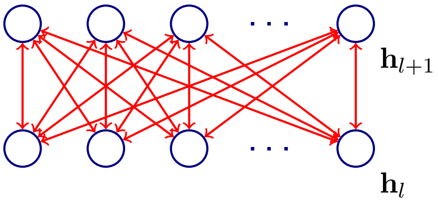


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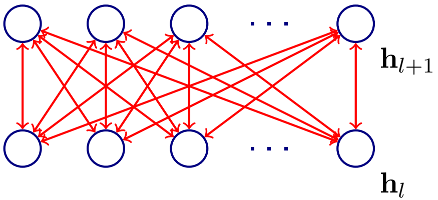


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- The conditional probability:

$$P(\mathbf{h}|\mathbf{x}) = \frac{\exp\left(\mathbf{b}^T\mathbf{x} + \mathbf{c}^T\mathbf{h} + \mathbf{h}^T W\mathbf{x}\right)}{\sum_{\tilde{\mathbf{h}}} \exp\left(\mathbf{b}^T\mathbf{x} + \mathbf{c}^T\tilde{\mathbf{h}} + \tilde{\mathbf{h}}^T W\mathbf{x}\right)} = \prod_i P(\mathbf{h}_i|\mathbf{x})$$

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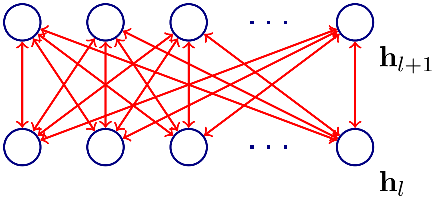


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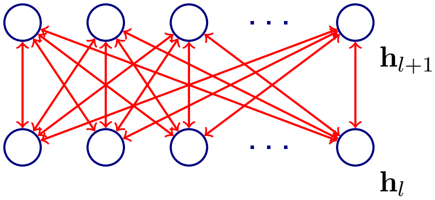


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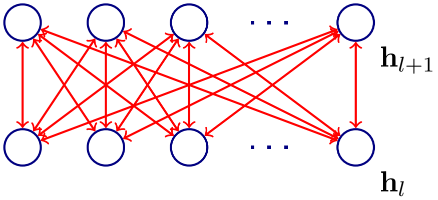


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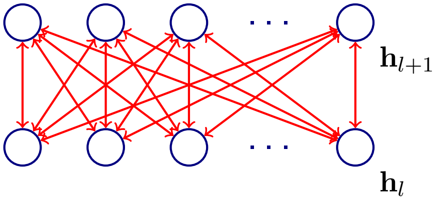


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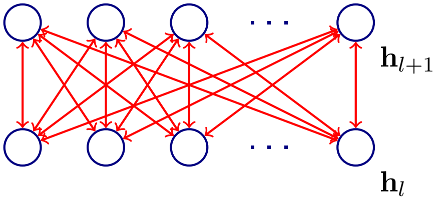


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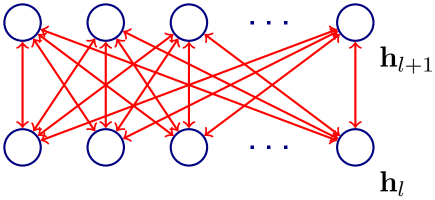


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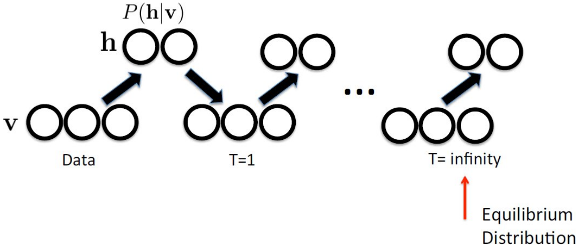
# Approximate Learning, Alternating Gibbs Sampling

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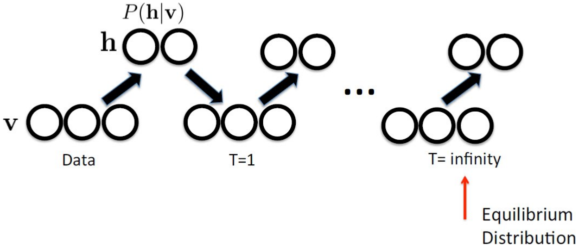


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- Aside: Easy to extend RBM (and contrastive divergence) to the continuous case

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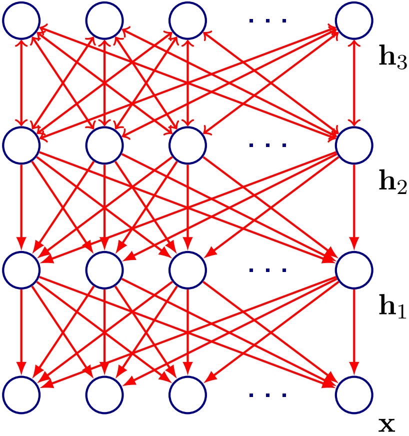
- Originally proposed by Hinton and Sejnowski (1983)
- Important historically. But very difficult to train (why?)

# Back to Deep Belief Networks

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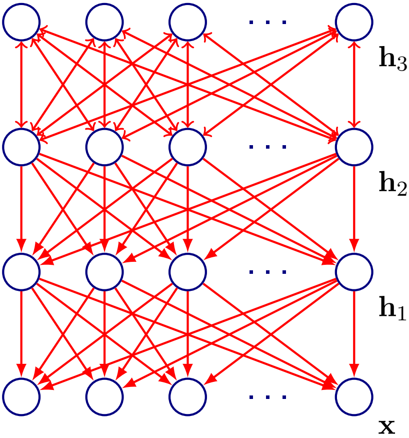


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- Can then be discriminatively fine-tuned using backpropagation

# Deep Autoencoders (2006)

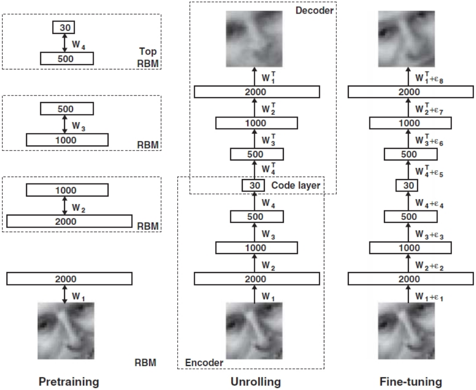
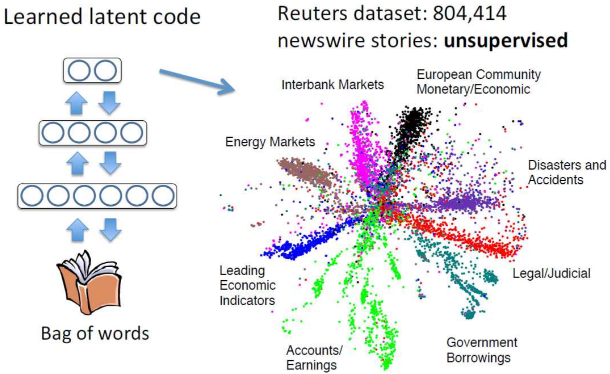



**Fig. 1.** Pretraining consists of learning a stack of restricted Boltzmann machines (RBMs), each having only one layer of feature detectors. The learned feature activations of one RBM are used as the "data" for training the next RBM in the stack. After the pretraining, the RBMs are "unrolled" to create a deep autoencoder, which is then fine-tuned using backpropagation of error derivatives.

G. E. Hinton, R. R. Salakhutdinov, Reducing the dimensionality of data with neural networks, Science, 2006

From last time: Was hard to train deep networks from scratch in 2006!

# Semantic Hashing




G. Hinton and R. Salakhutdinov, "Semantic Hashing", 2006

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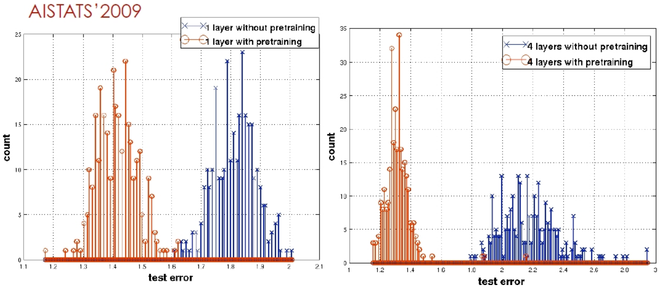
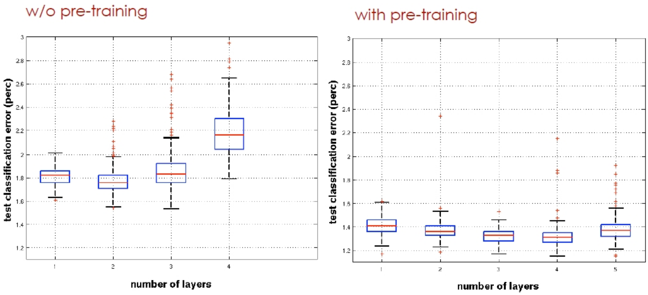
# Effect of Unsupervised Pre-training



# Effect of Unsupervised Pre-training



- Important topics we didn't talk about in detail/at all:
  - Joint unsupervised training of all layers (Wake-Sleep algorithm)
  - Deep Boltzmann Machines
  - Variational bounds justifying greedy layerwise training
  - Conditional RBMs, Multimodal RBMs, Temporal RBMs etc

# Next Time

- Some Applications of methods we just considered

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- Generative Adversarial Networks