Improved Approximation for Node-Disjoint Paths in Grids with Sources on the Boundary

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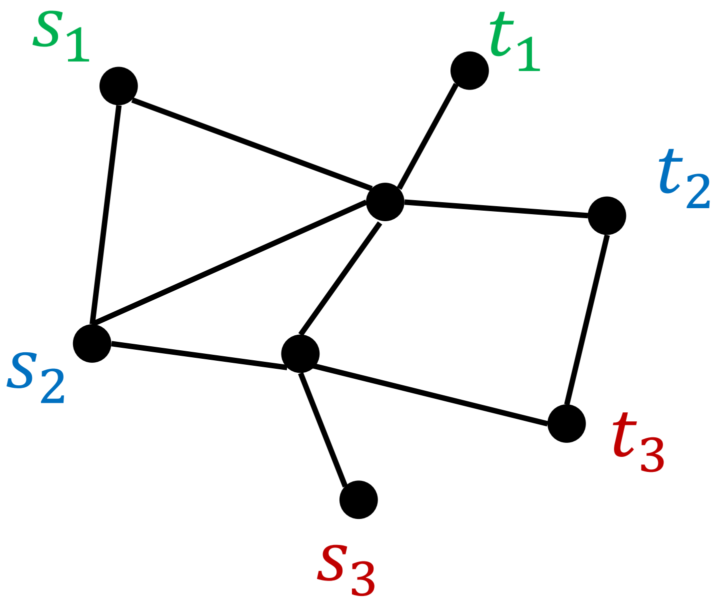
Node-Disjoint Paths (NDP)

Input: Undirected graph and demand pairs $(s_1, t_1), \dots, (s_k, t_k)$



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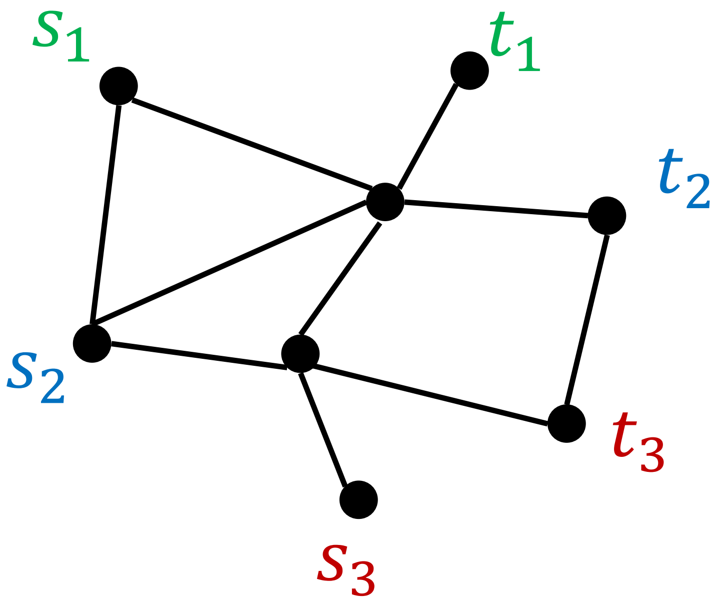

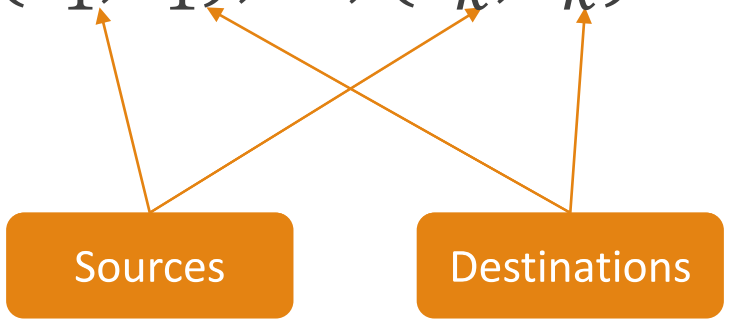

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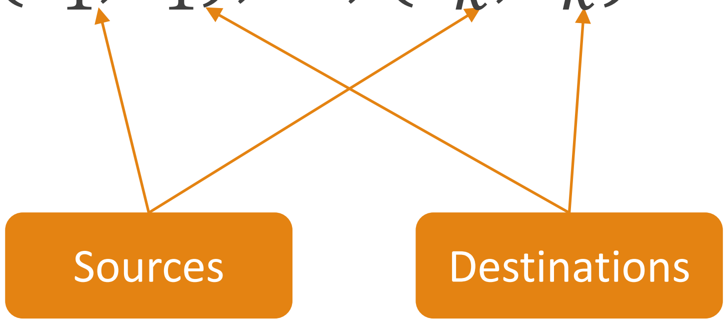

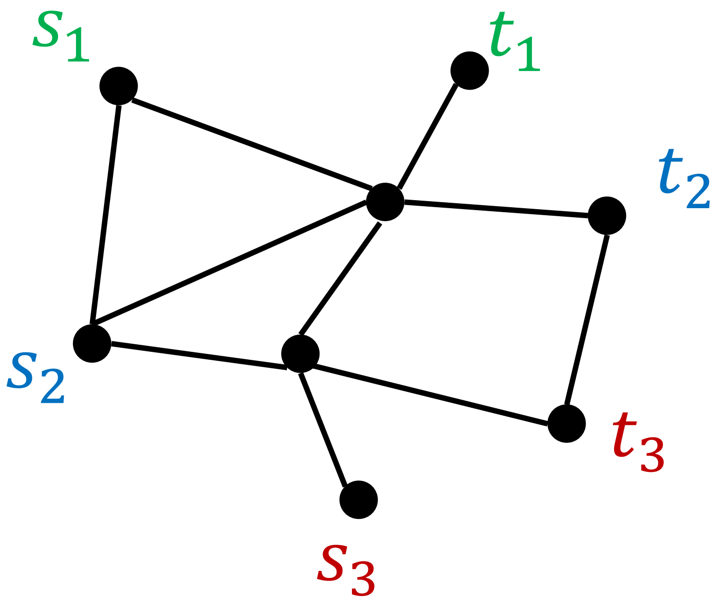

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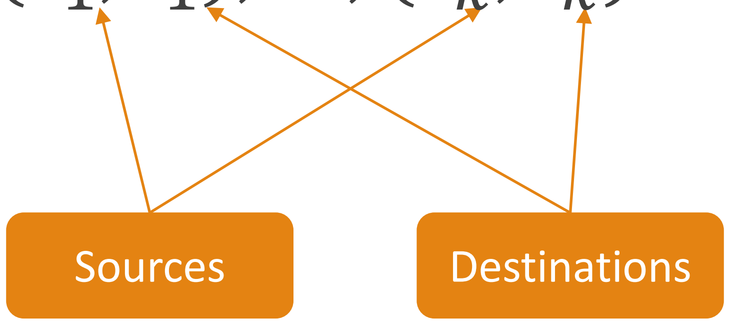

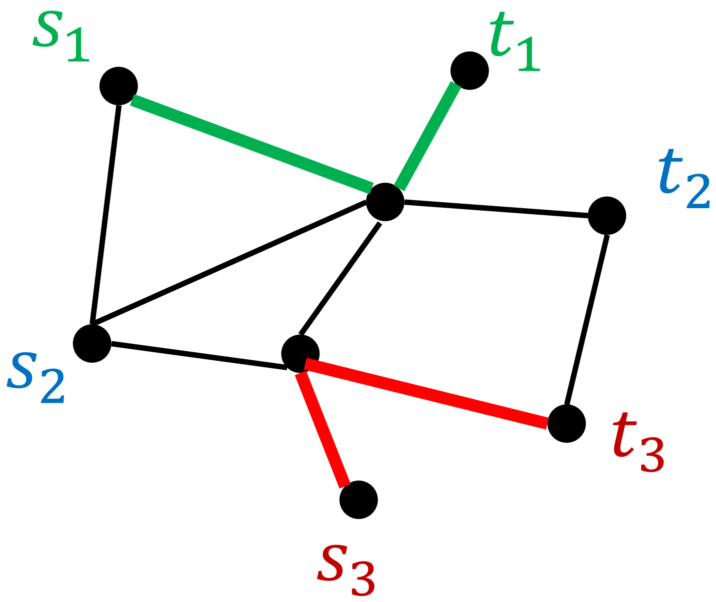

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  - FPT algorithm: $f(k) \cdot n^2$ [Robertson, Seymour '90 → Kawarbayashi, Kobayashi, Reed '12]

$$f(k) = 2^{2^{2^{2^{\cdot^{\cdot^{\cdot^{(k)}}}}}}}$$

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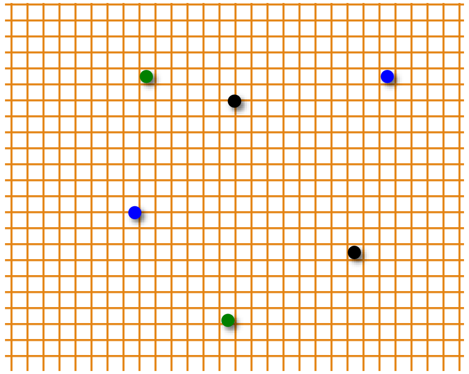
- What about *simpler* cases?
  - Analysis of $O(\sqrt{n})$-approx. algorithm is tight on grids! ☹

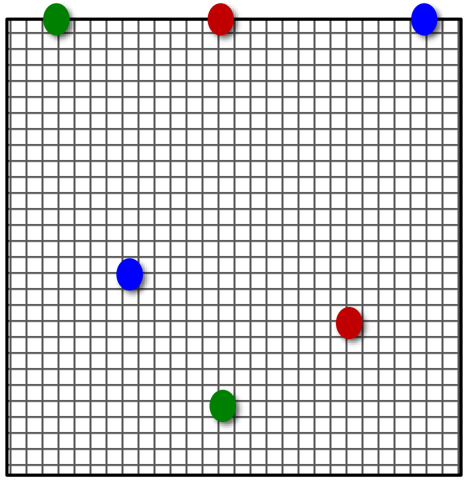
NDP-Grid

- $O(n^{1/4})$ −approx. for NDP-Grid [Chuzhoy, Kim '15]
- $n^{\Omega(1/(\log\log n)^2)}$ hardness [Chuzhoy, Kim, N. '18]



Our Result

$2^{\tilde{O}(\sqrt{\log n})}$-approximation algorithm for NDP-Grid if sources appear on the boundary



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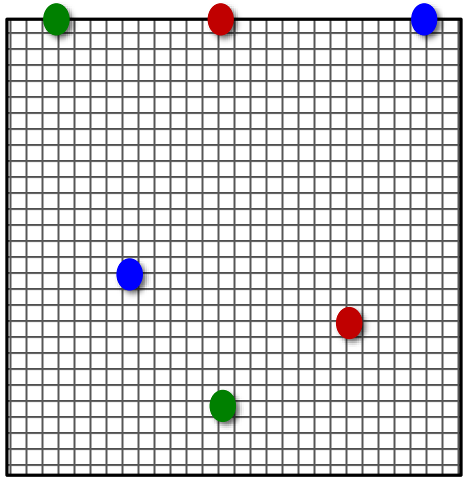
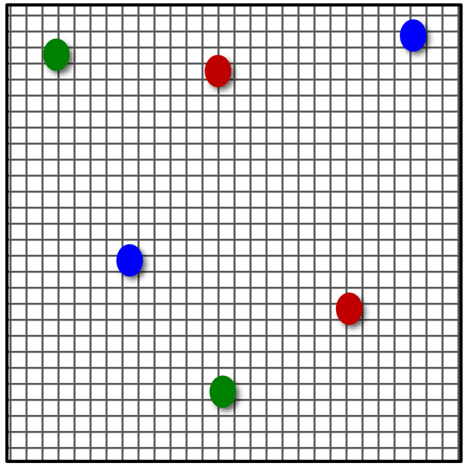
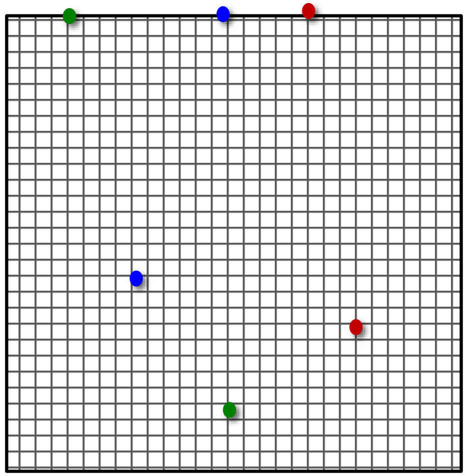


$\delta \cdot 2^{\tilde{O}(\sqrt{\log n})}$-approximation algorithm for NDP-Grid if sources are at distance $\leq \delta$ from the boundary



"Complementary" Results



$2^{O(\sqrt{\log n})}$ −approximation

Grid with sources on boundary

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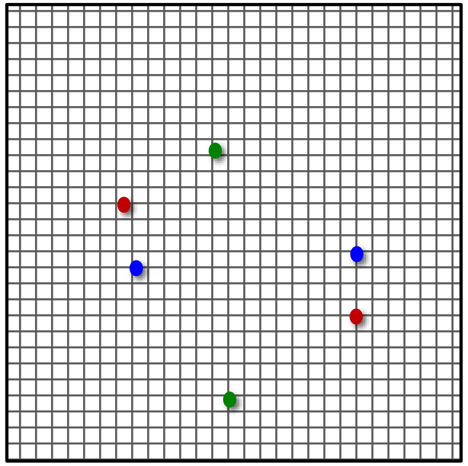

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Grid with sources far from boundary

[Chuzhoy, Kim, N. '18]

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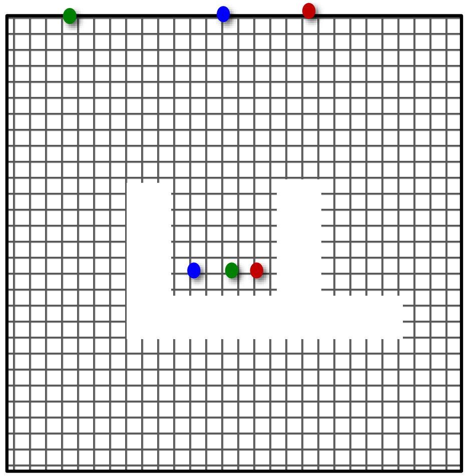

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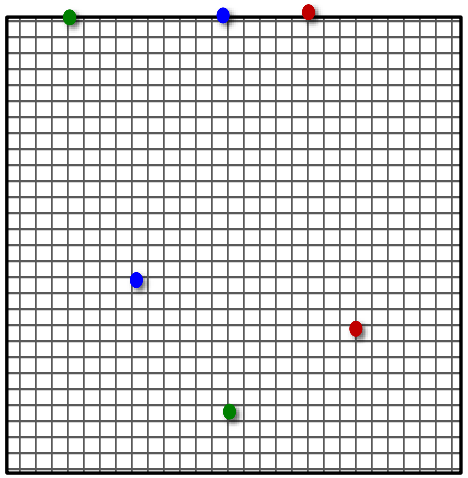
Grid with holes

[Chuzhoy, Kim, N. ‘17]



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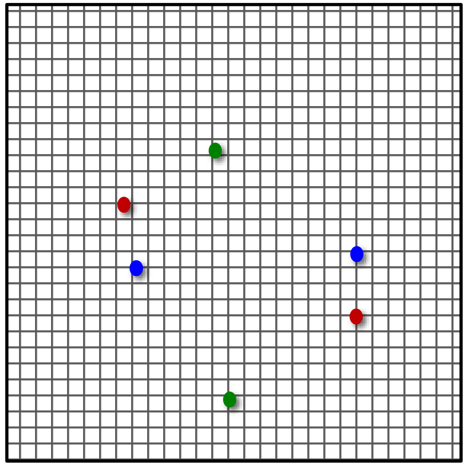

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On grid with sources and destinations on boundary, integrality gap is $\Omega(\sqrt{n})$ ☹

Multicommodity Flows: Is That It?

- On grid with sources and destinations close to boundary, integrality gap is $\Omega(\sqrt{n})$ ☹
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- Even when sources and destinations are far boundary, integrality gap remains $\Omega(n^{1/8})$ ☹ [Chuzhoy, Kim '15]

Beyond Multicommodity Flows

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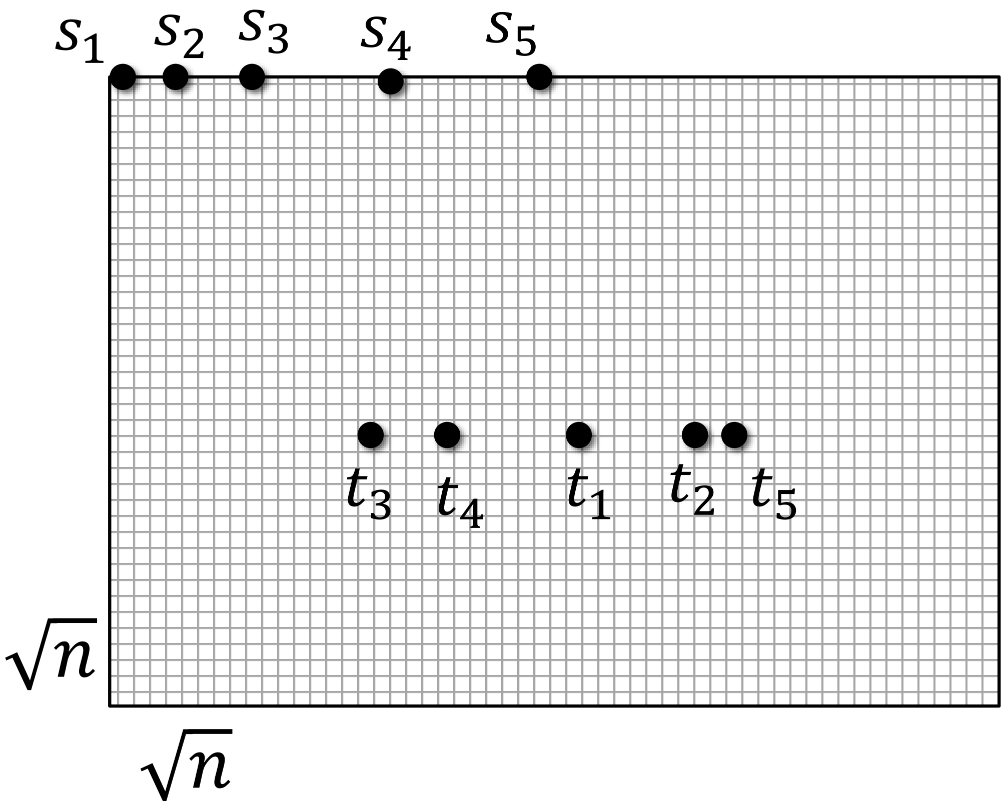
Assume for simplicity:

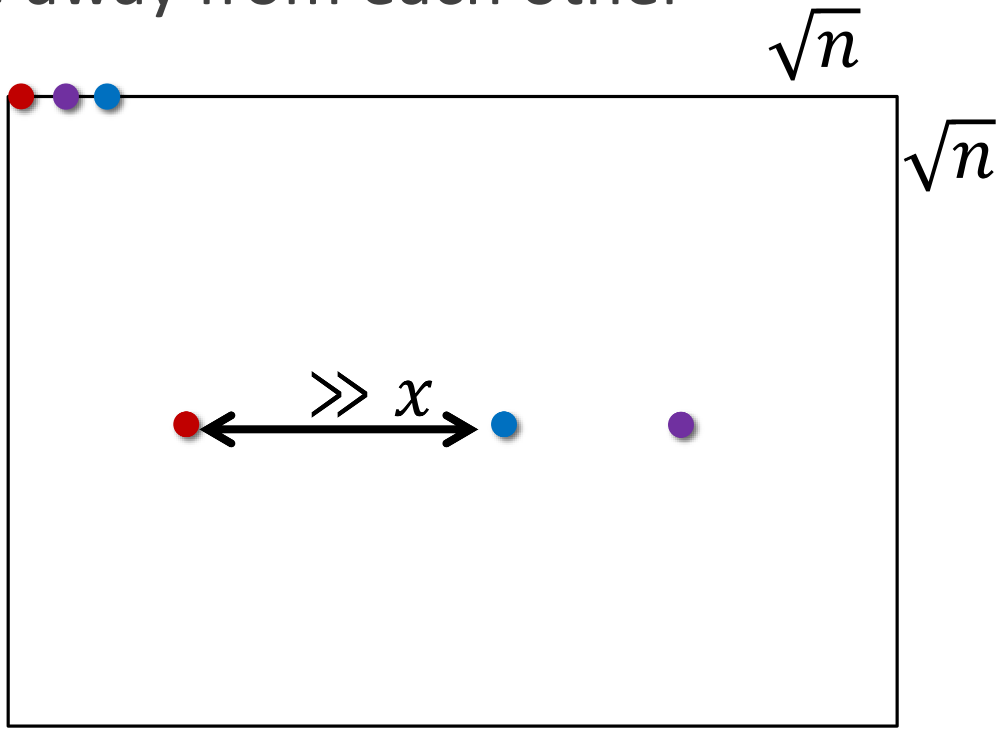
- All sources and destinations are distinct
- All sources lie on top boundary
- All destinations lie on a ***single row*** at distance $\gg OPT$ from grid boundaries



Routing

Assume we have:

- x demand pairs
- Their destinations are at $\gg x$ distance away from each other



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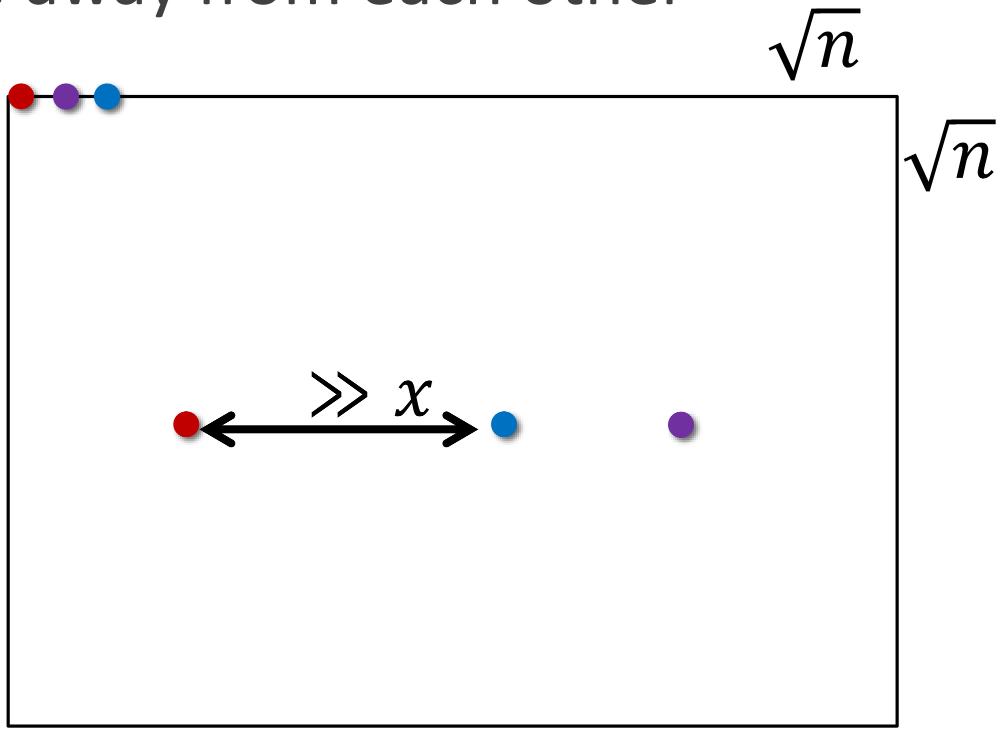
Snake-like Routing

Used in [Chuzhoy, Kim '15] and [Cutler, Shiloach '78]



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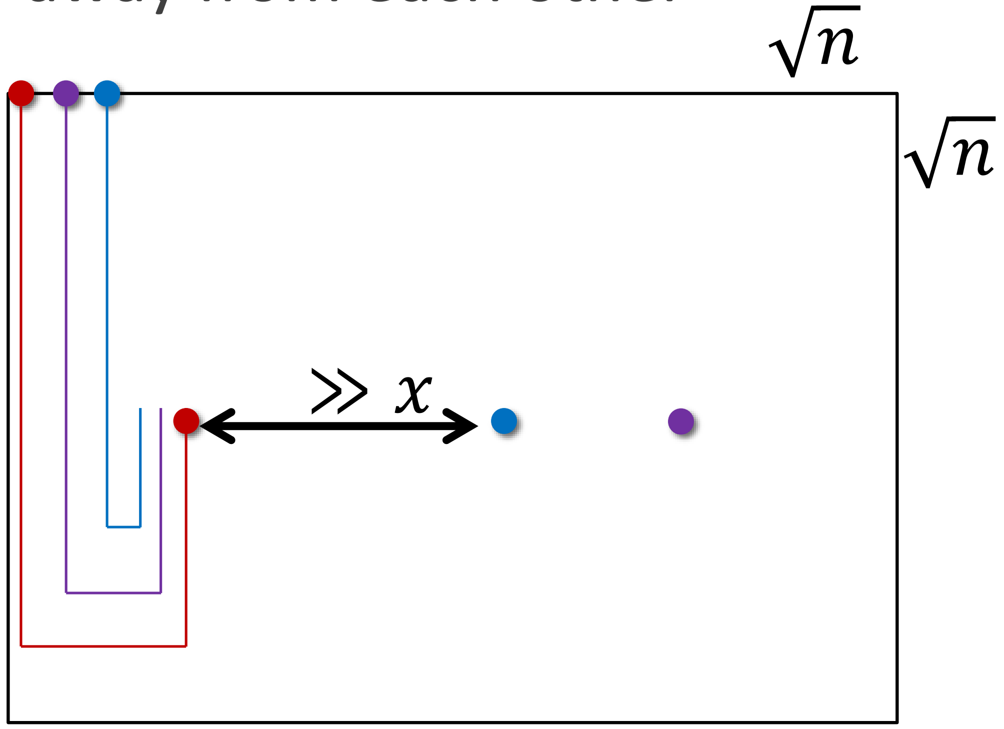

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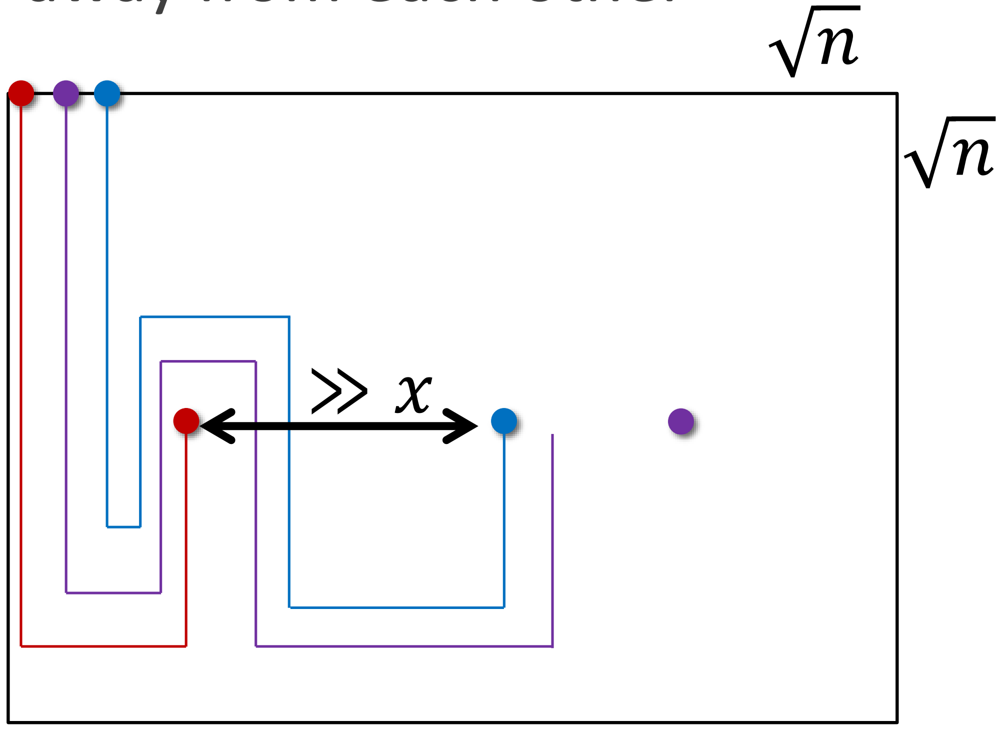

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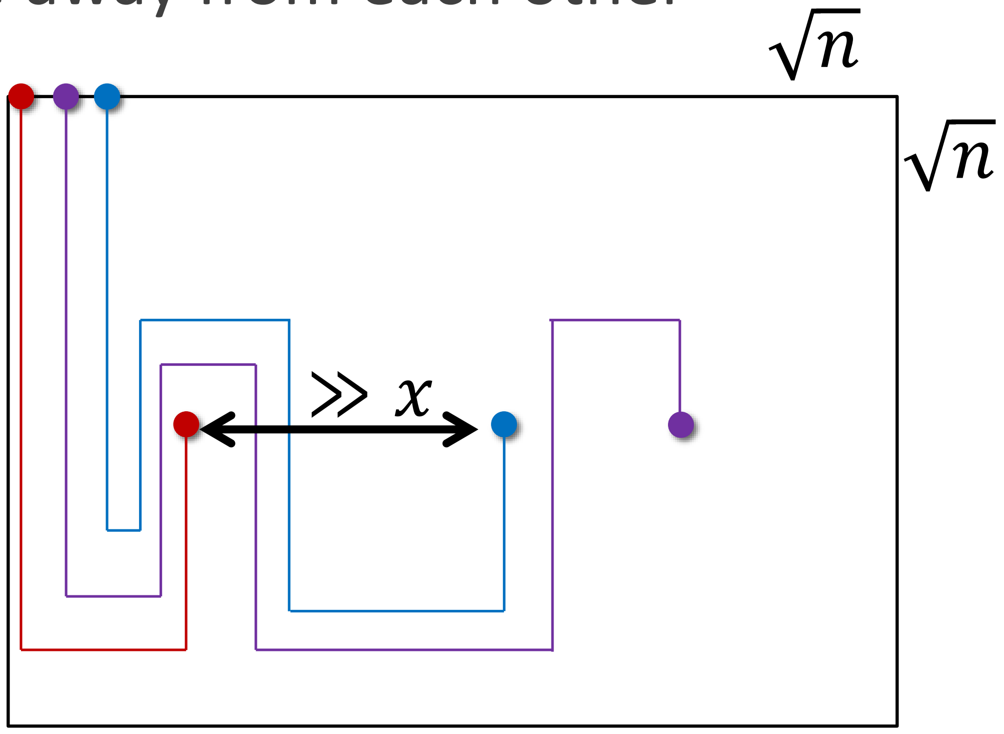

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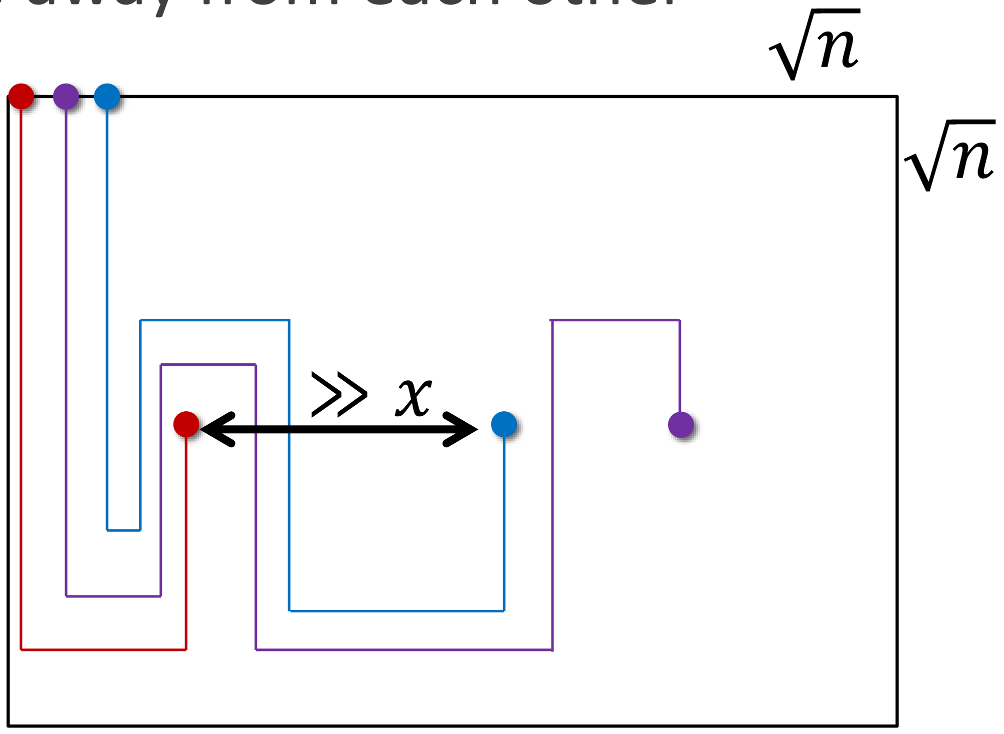
- Can't route more than $n^{1/4}$ demand pairs
- But OPT can be $\approx \sqrt{n}$

Looks very inefficient…



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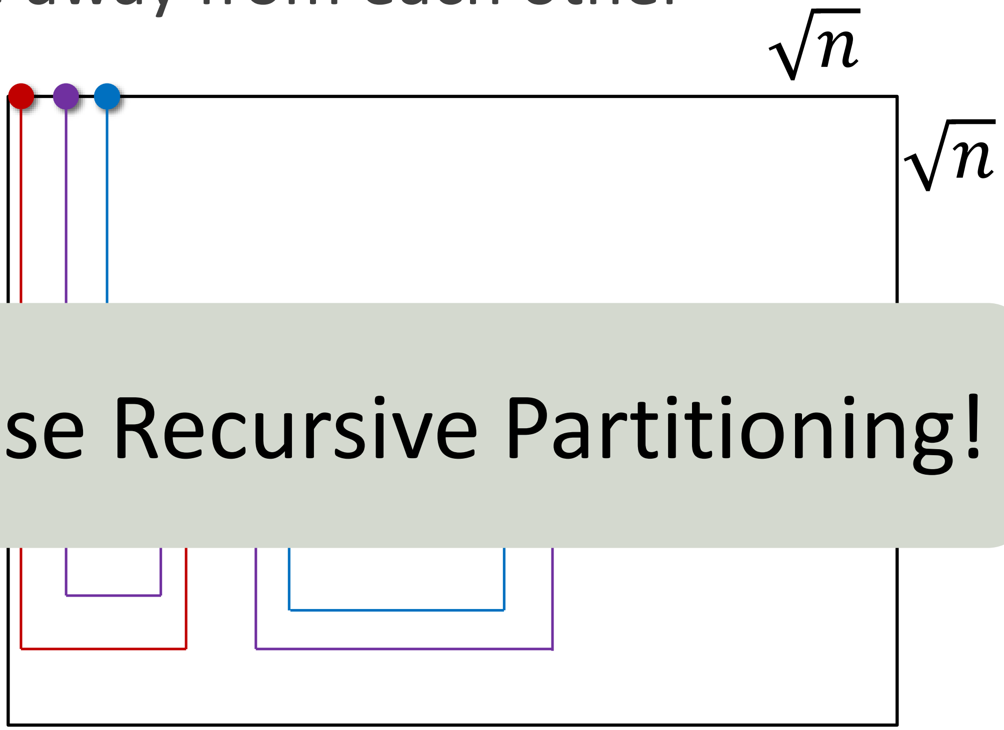

Use Recursive Partitioning!

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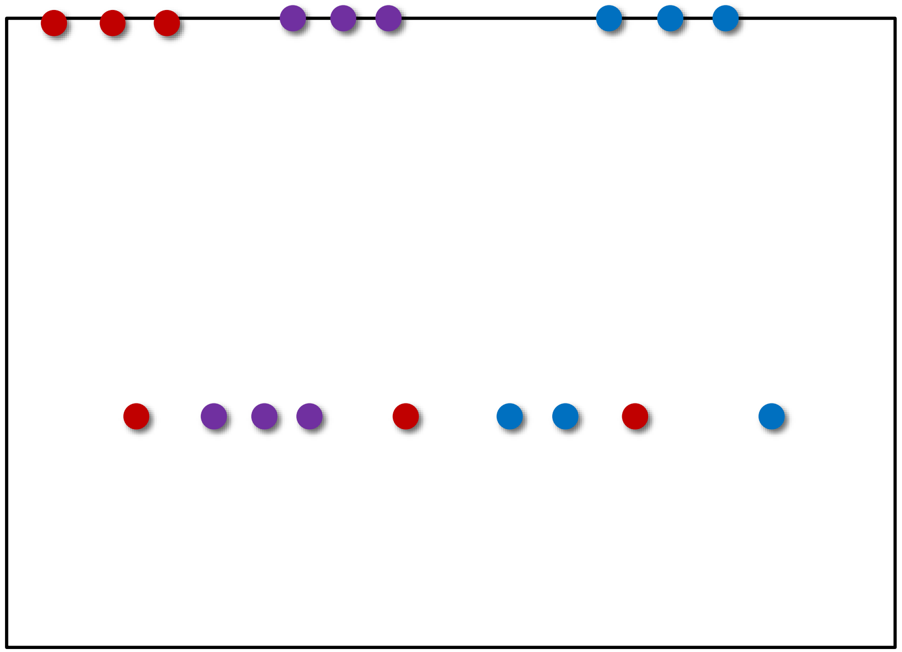
Def^n: A set of x demand pairs and its coloring by η colors:



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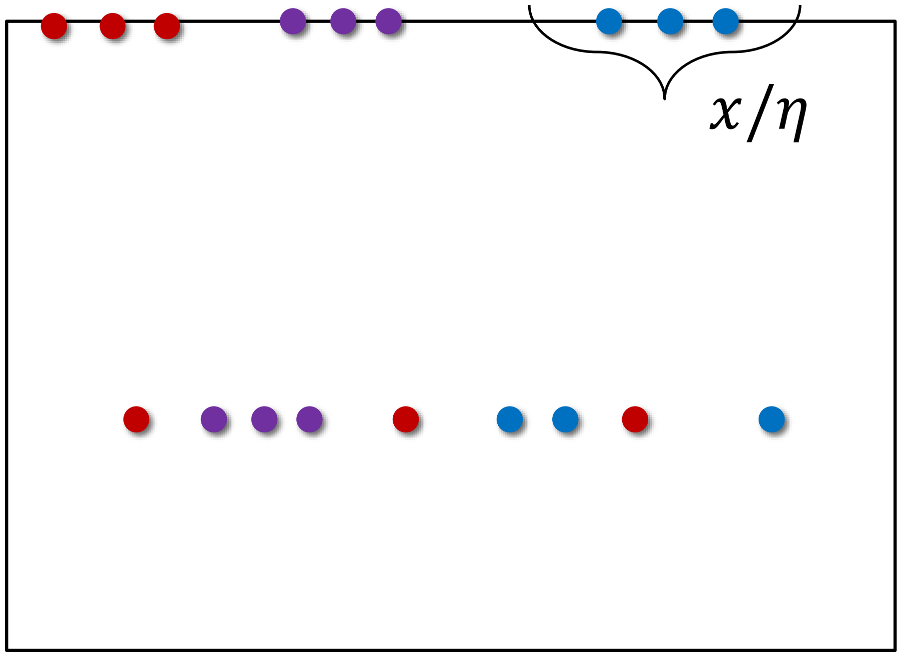

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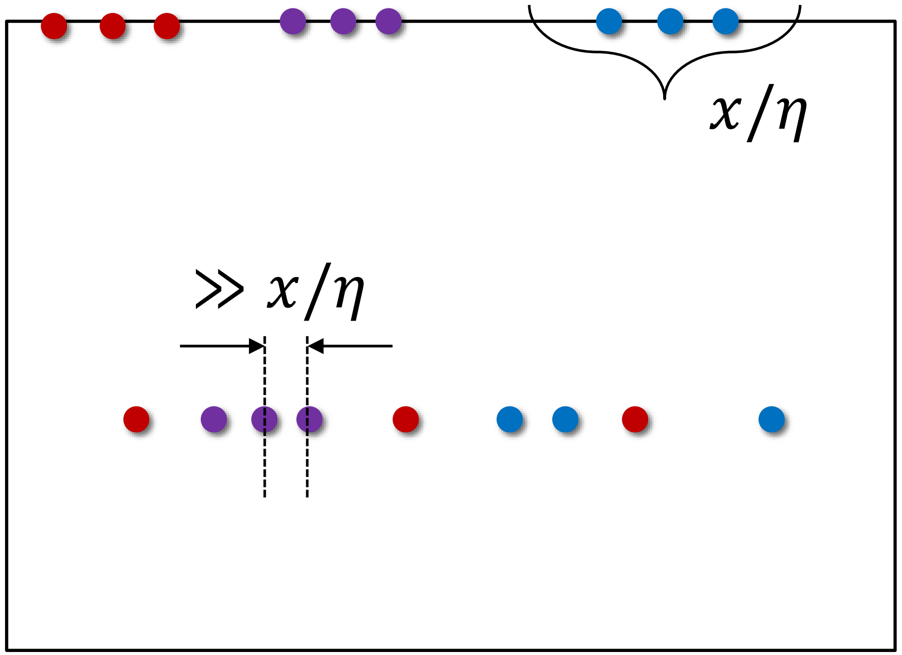

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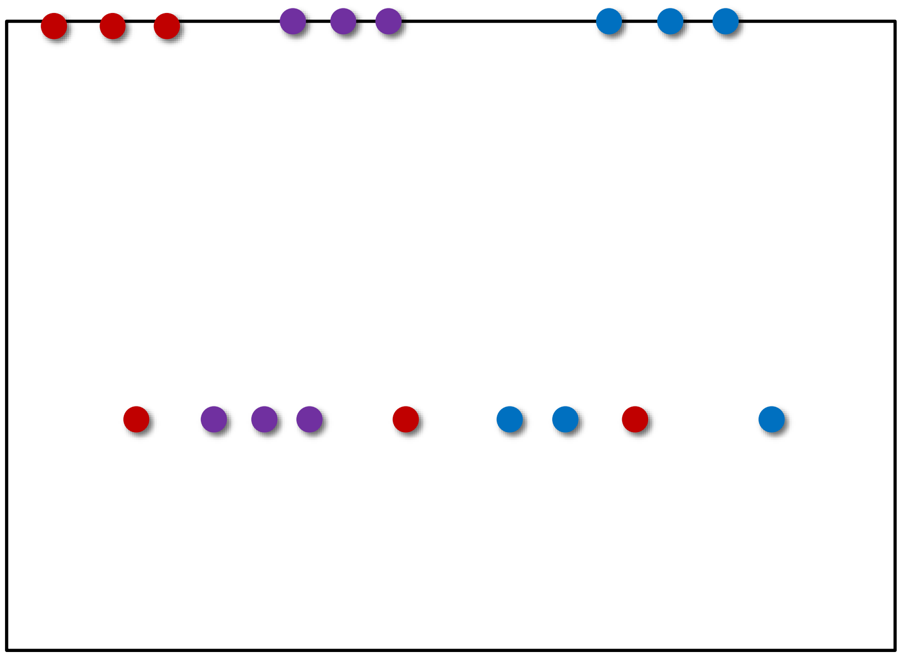

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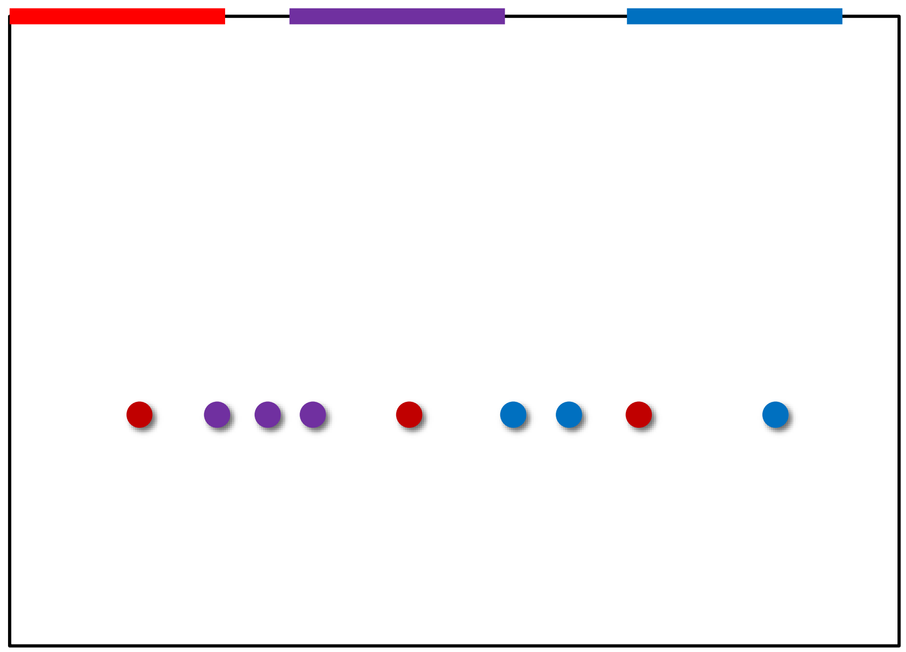

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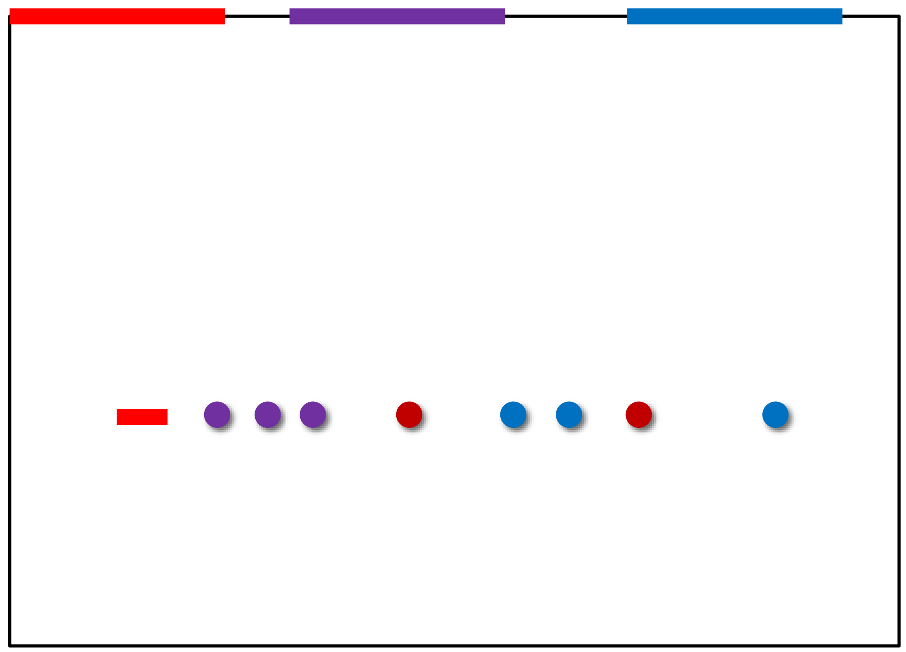

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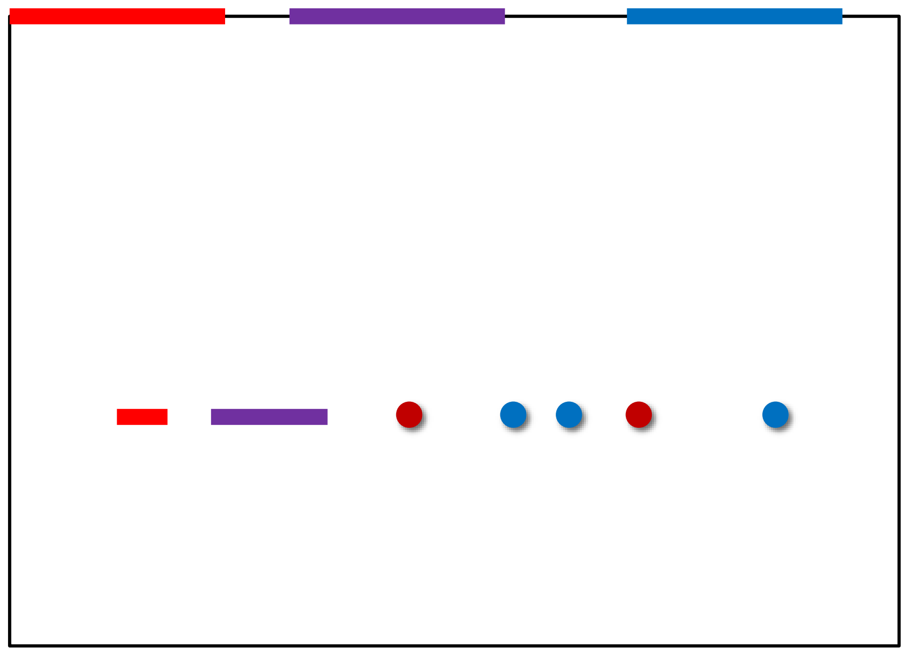

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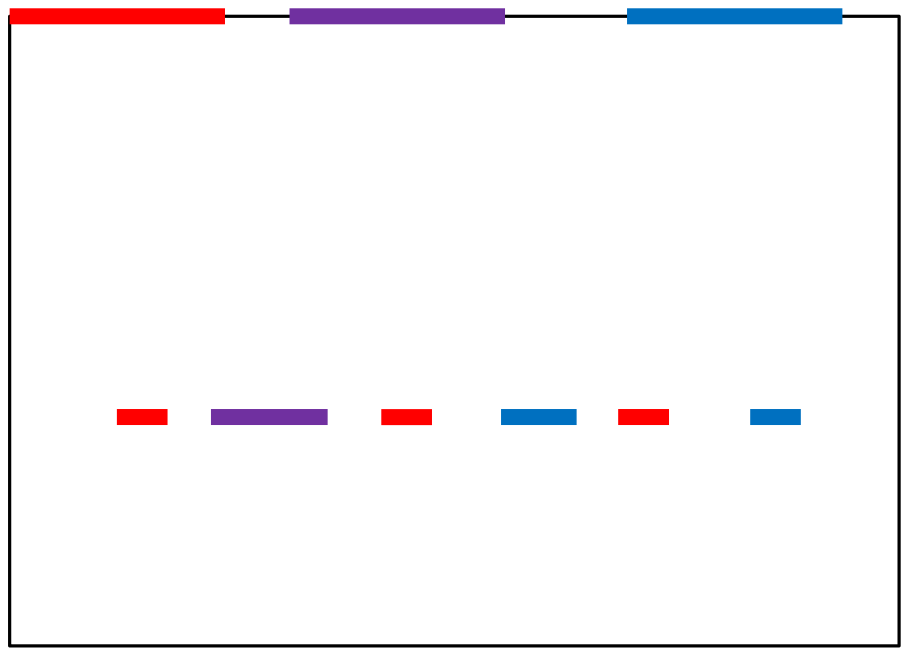

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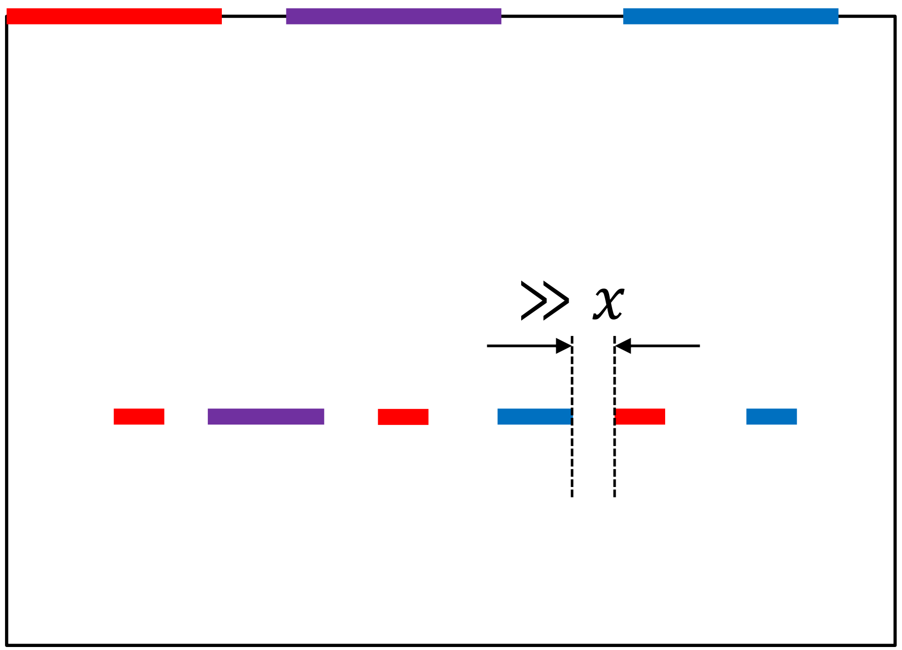
- Destination-intervals are at distance $\gg x$ from each other

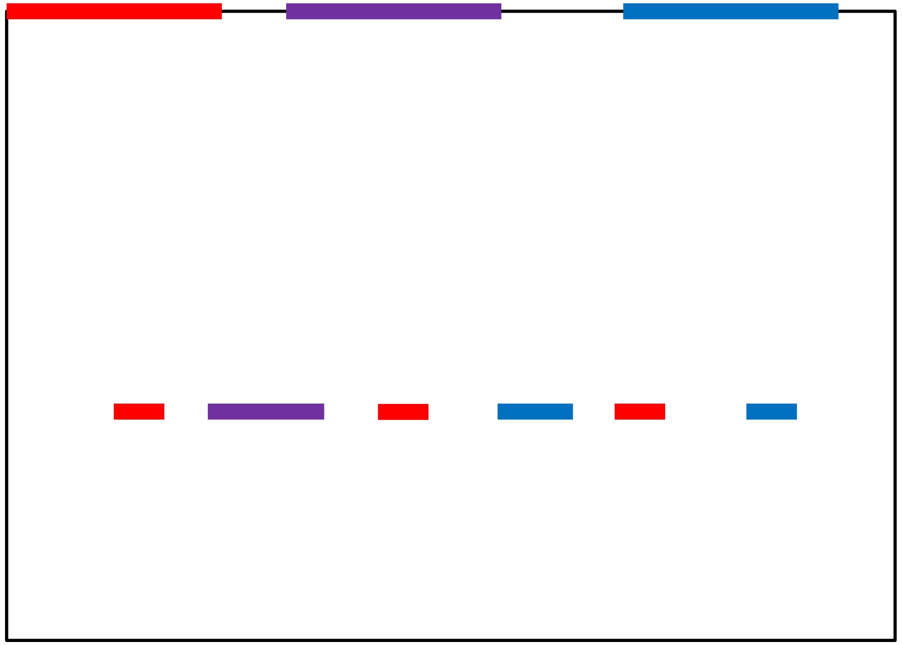


Recursive Partitioning

Theorem: Recursive Partitioning Property holds for x demand pairs $\Rightarrow$ can route all x demand pairs

Routing in two parts

- "Global Routing"
- "Local Routing"



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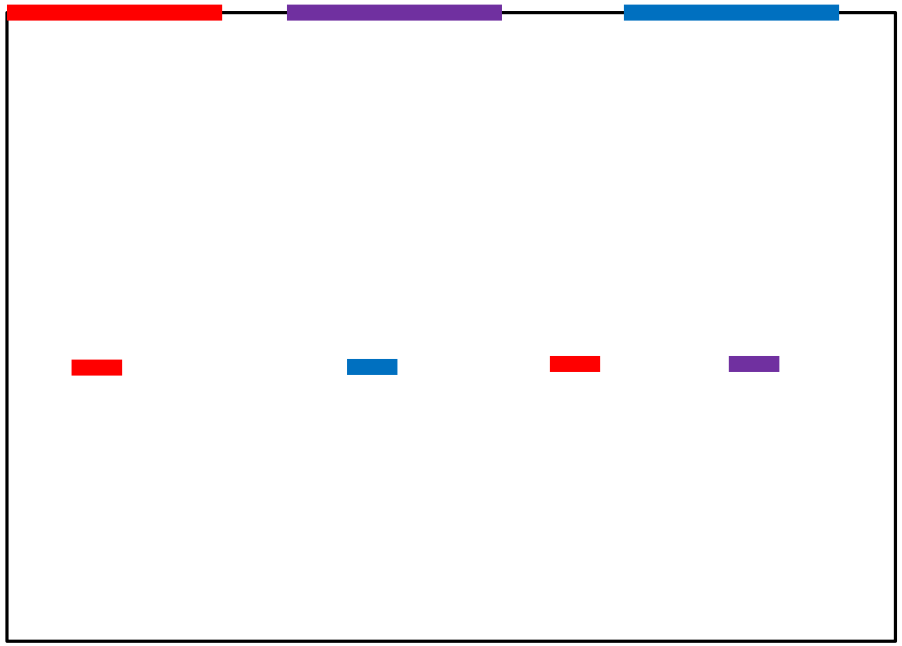

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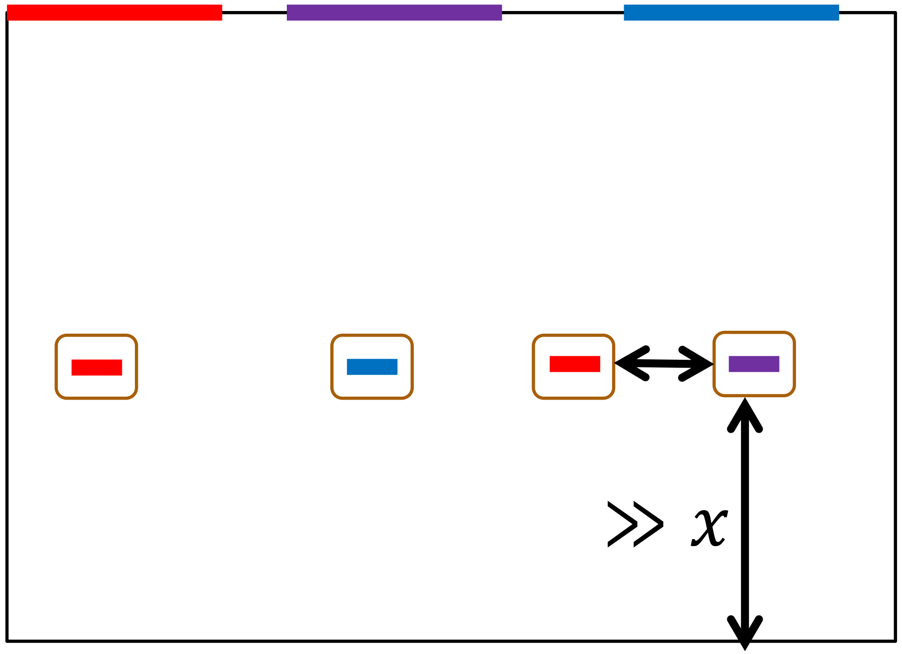

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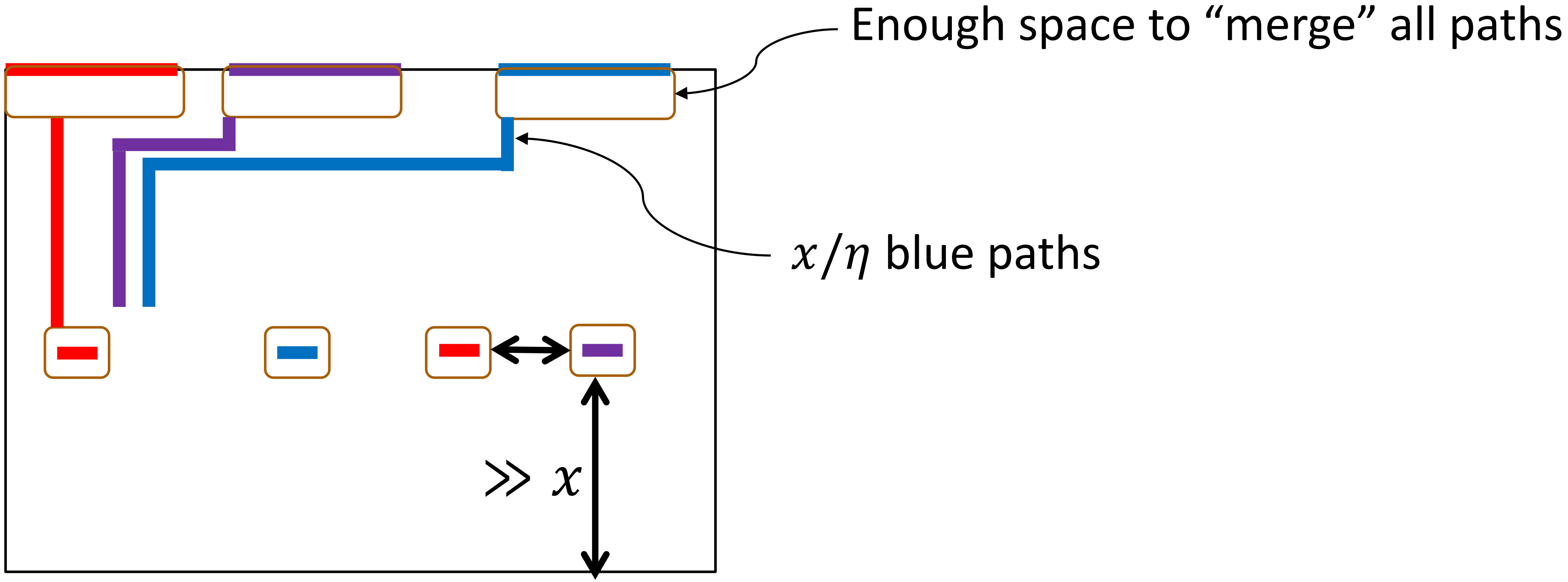

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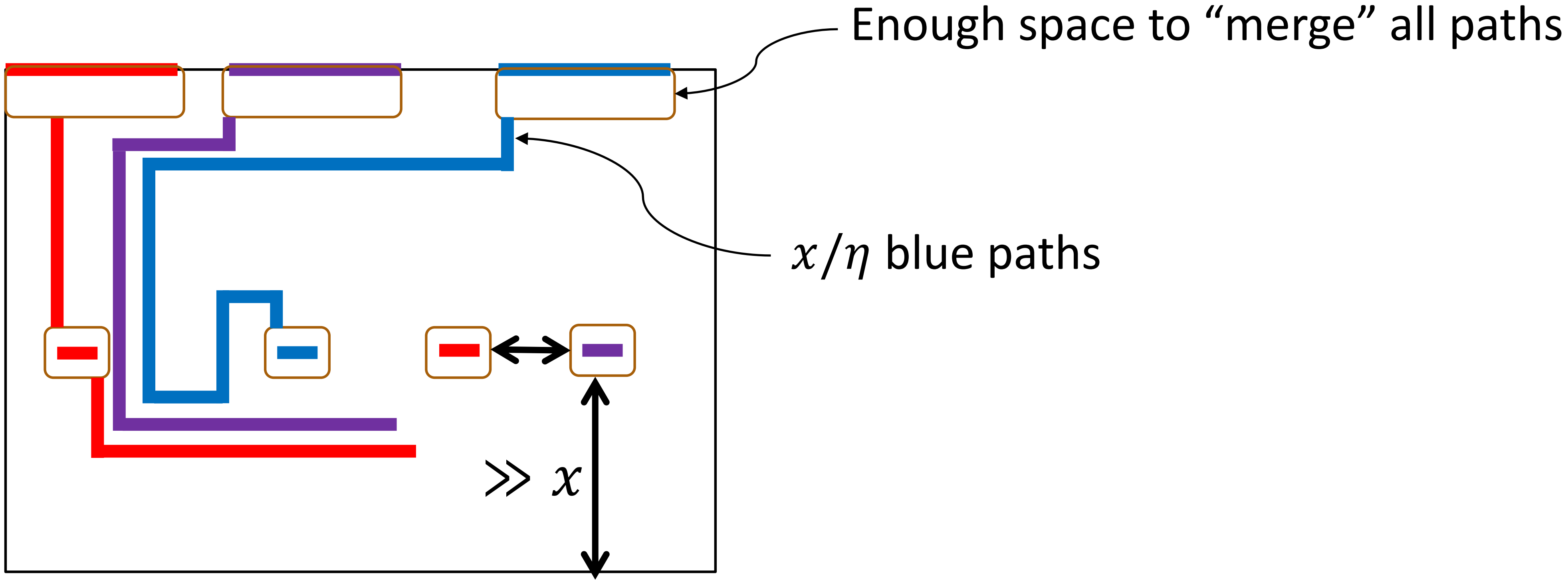

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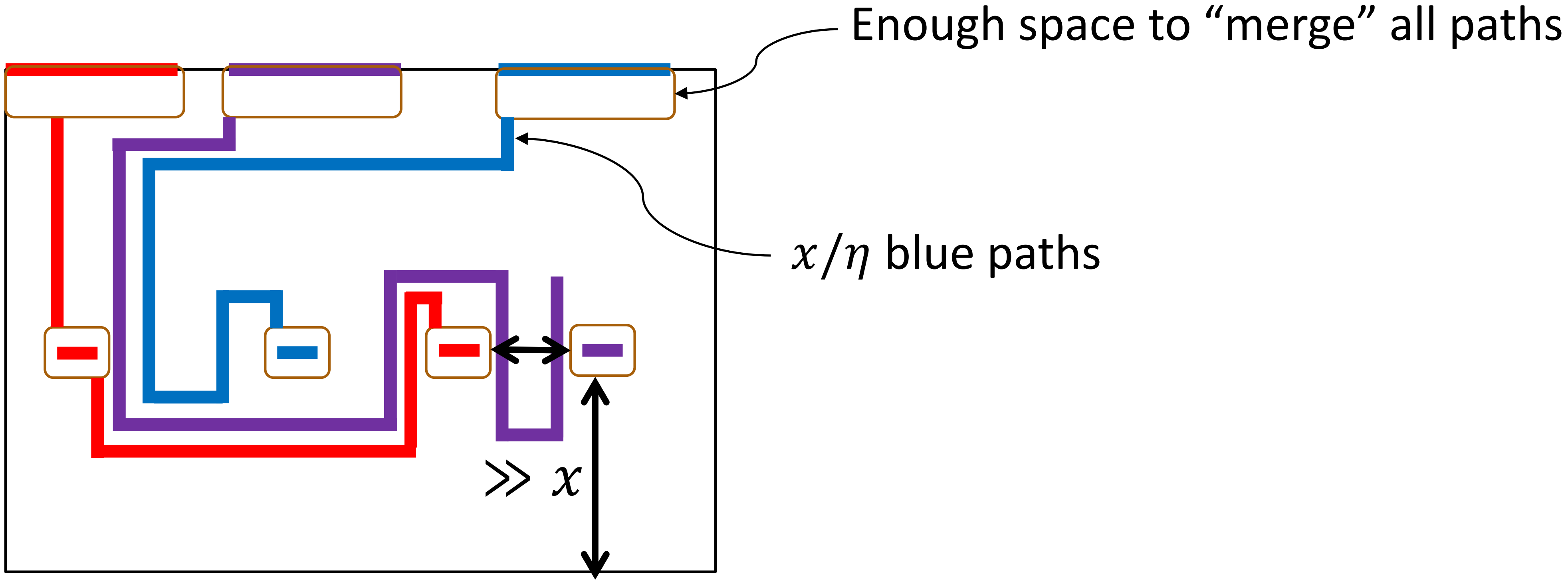

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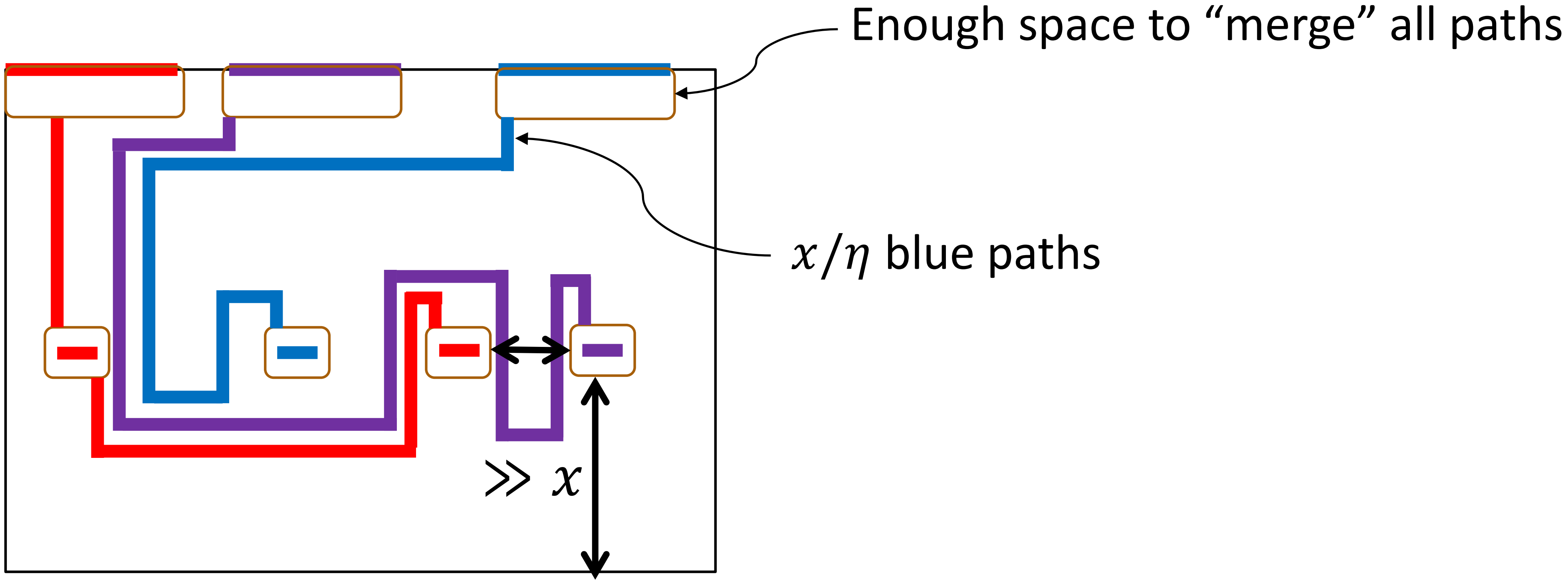

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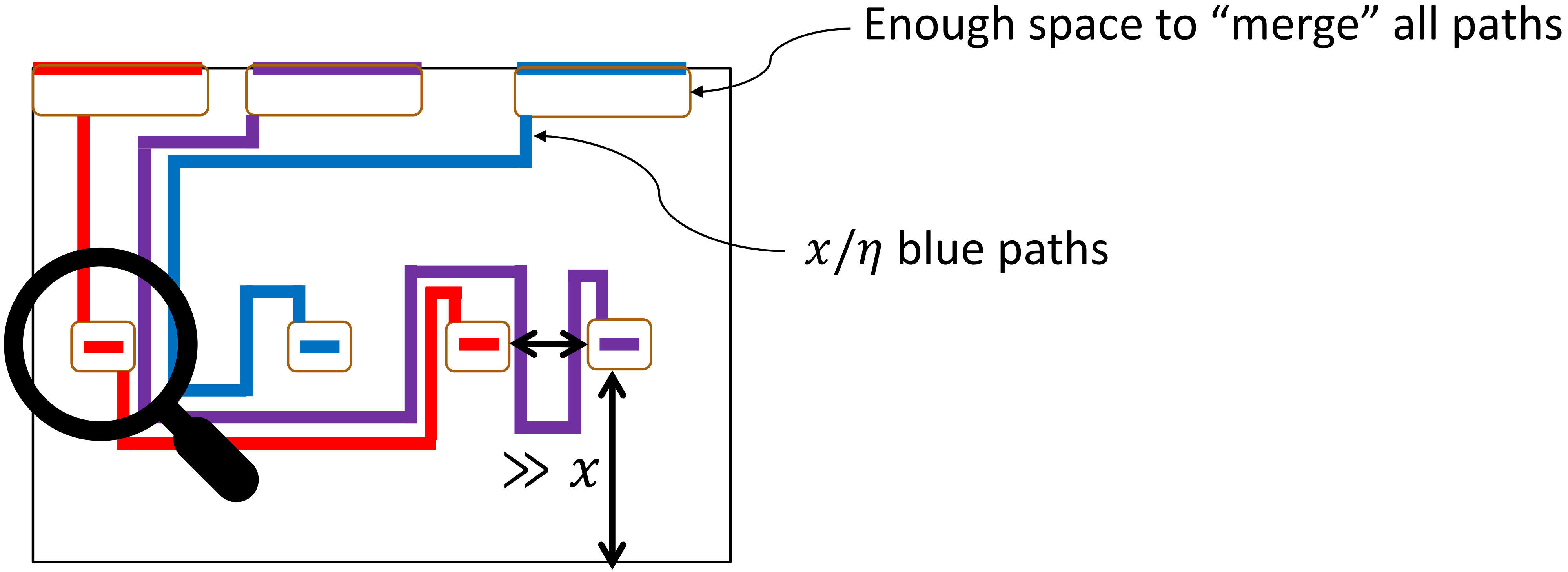
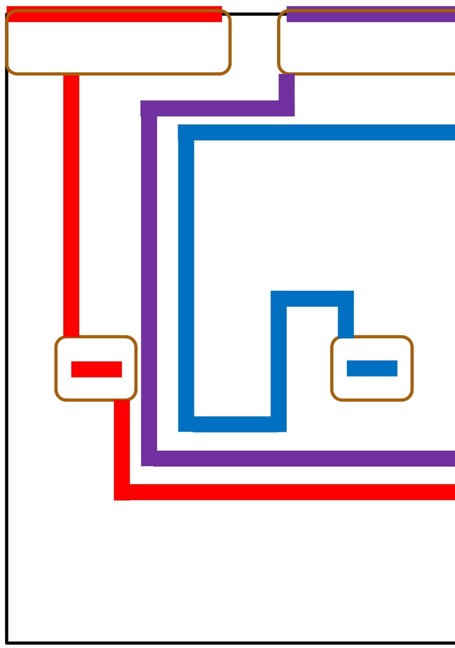


Part 2 | Local Routing



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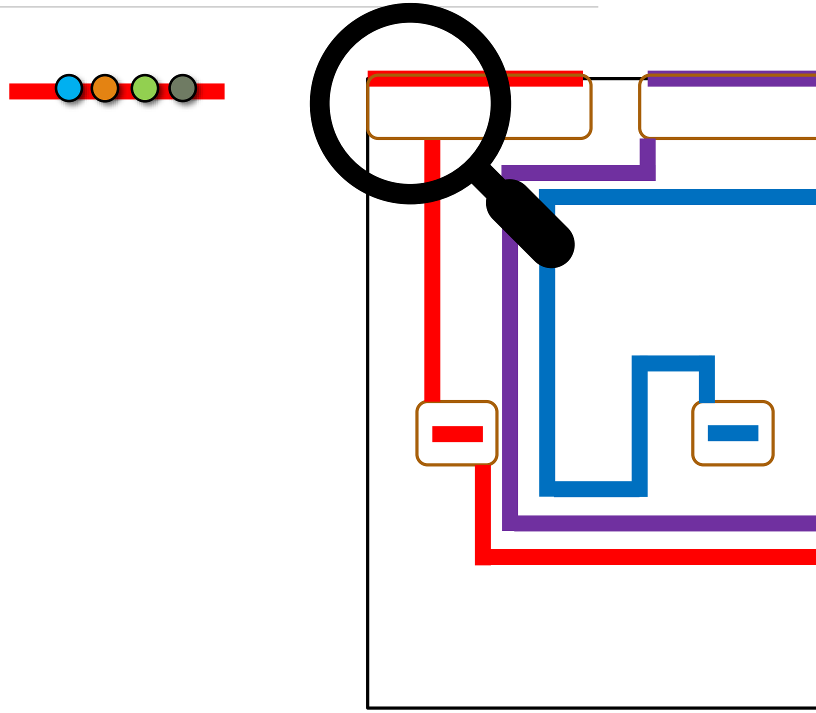

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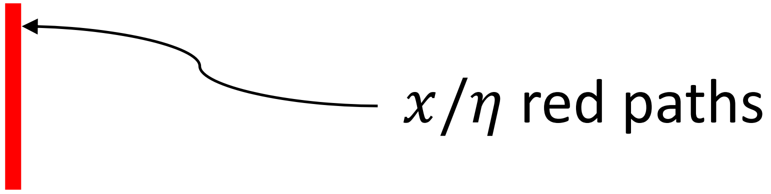

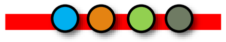

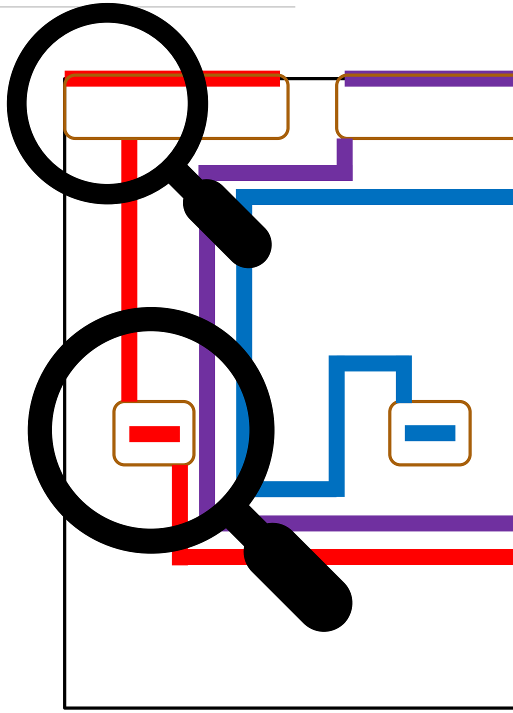

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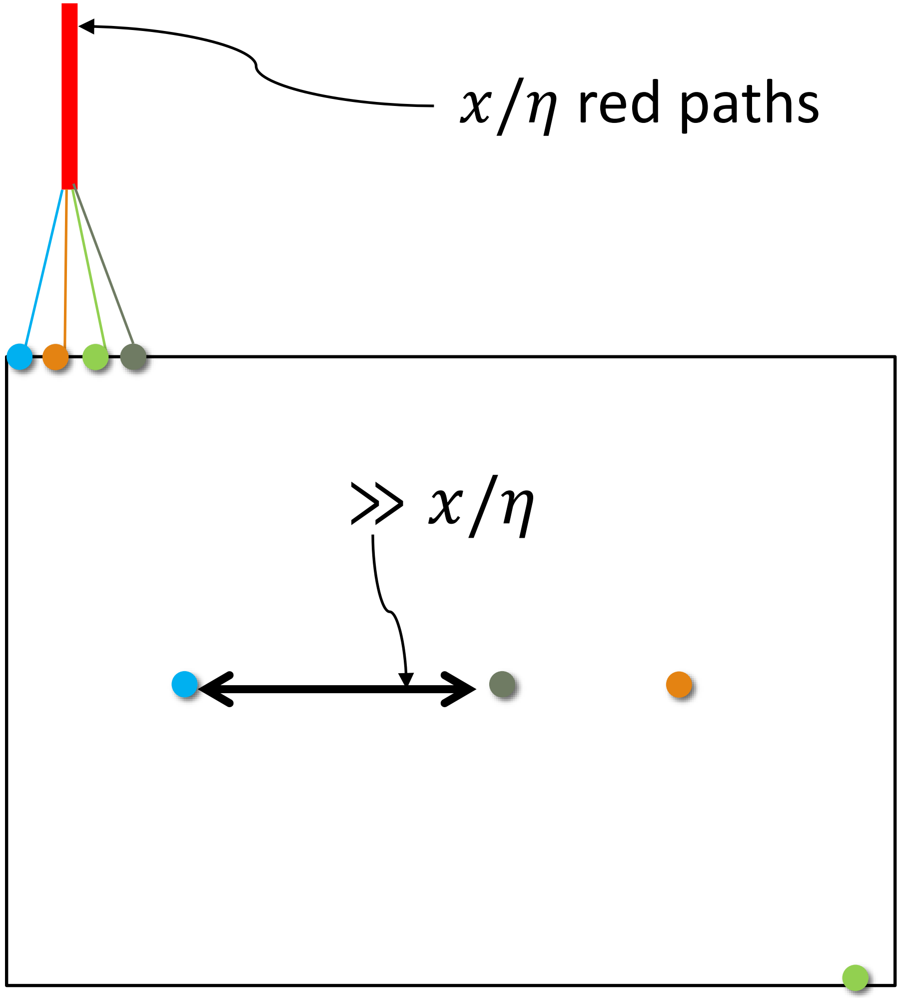

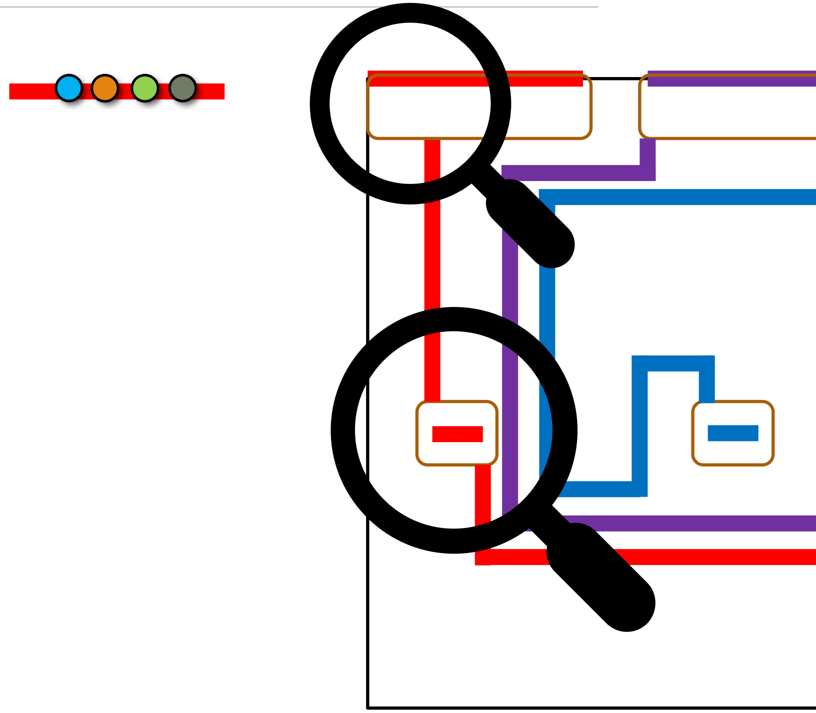

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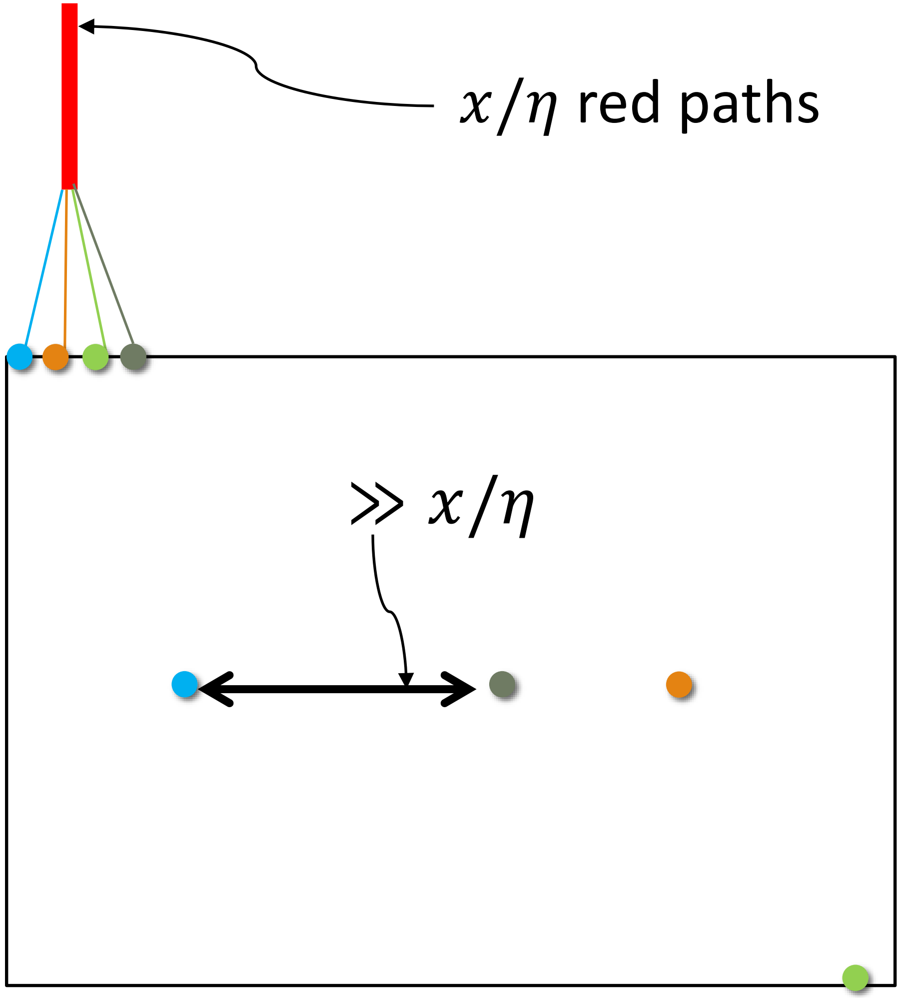

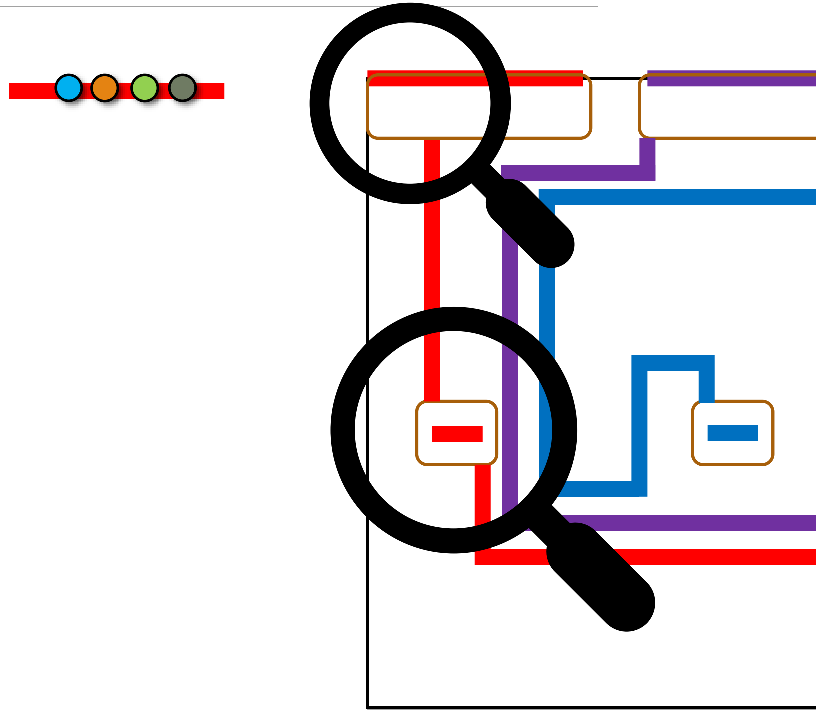

Snake-like routing
that we saw earlier!

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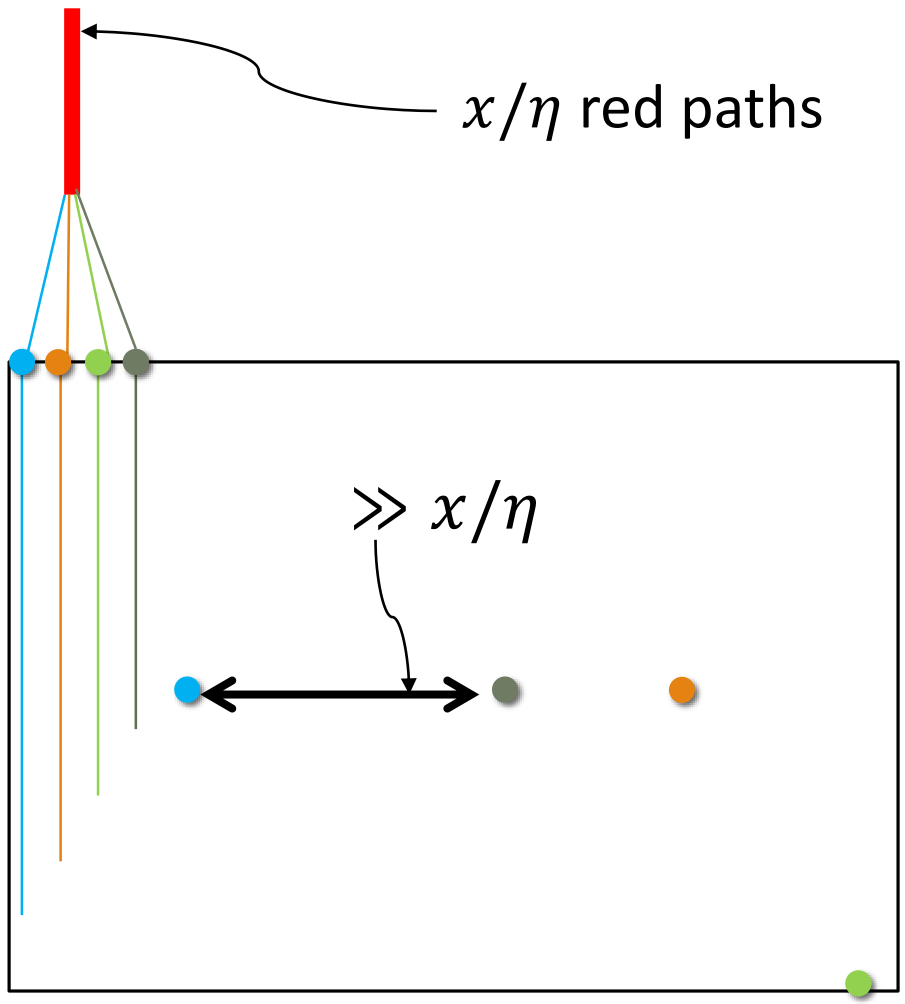

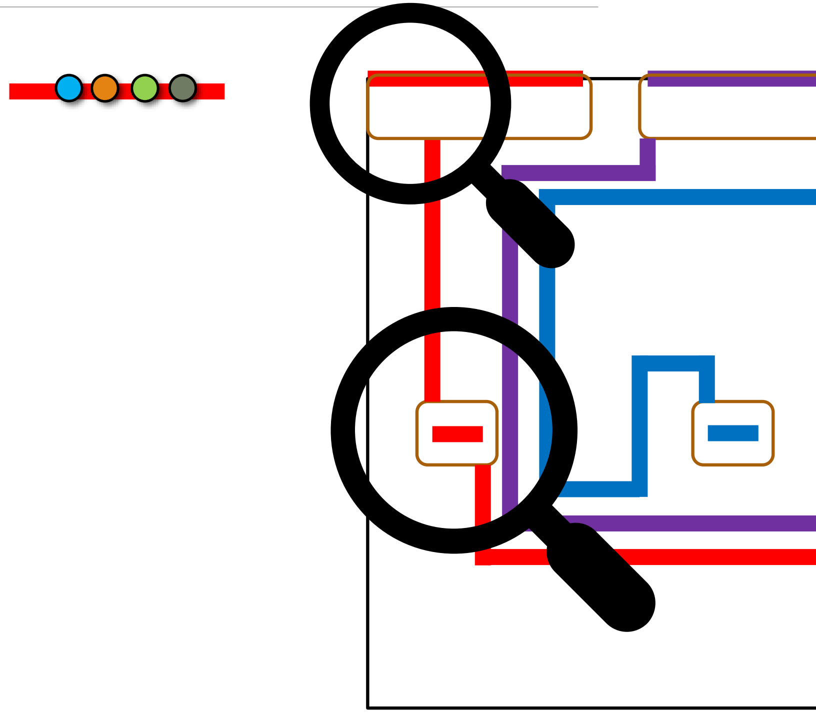

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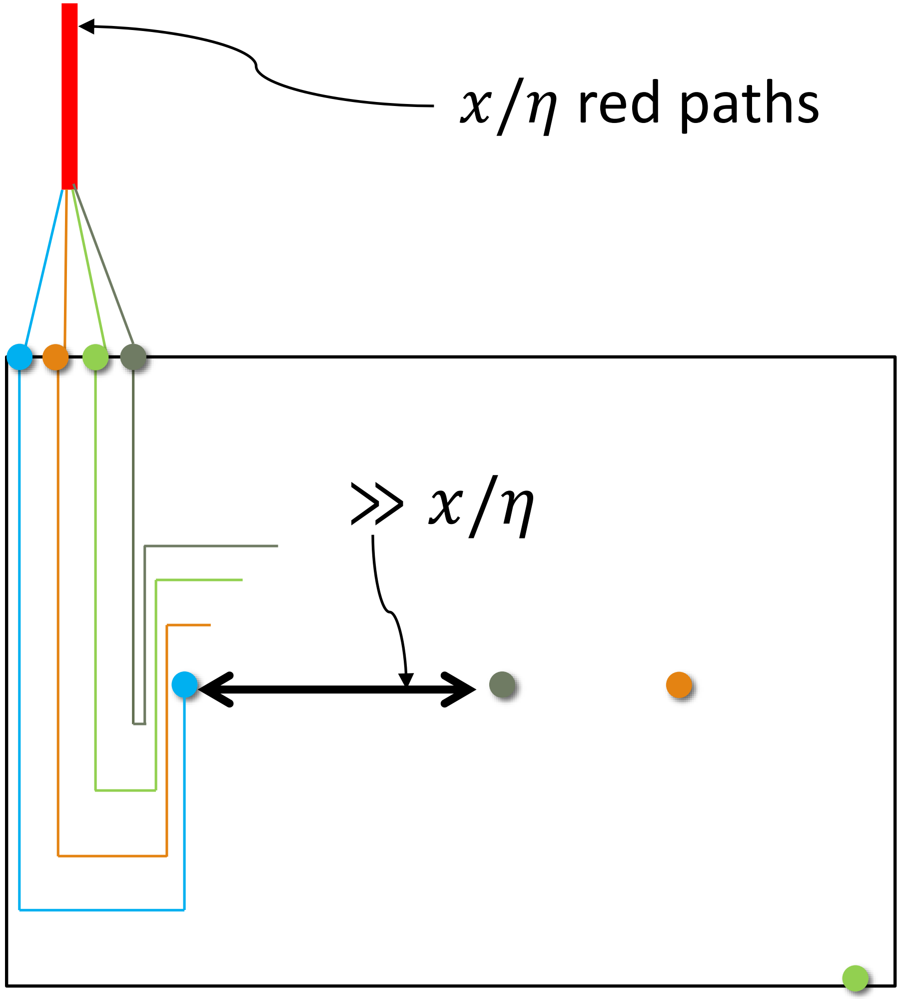

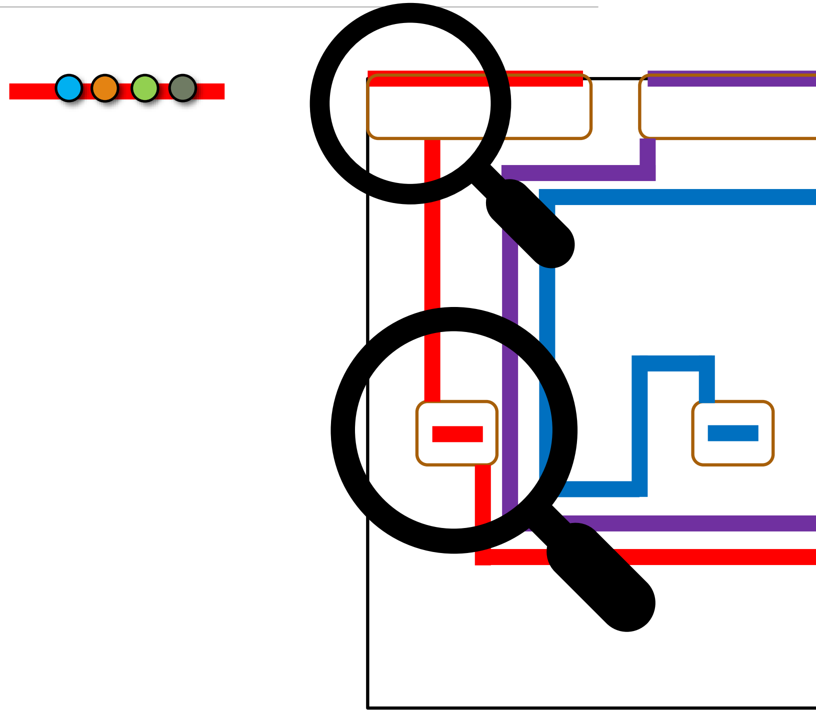

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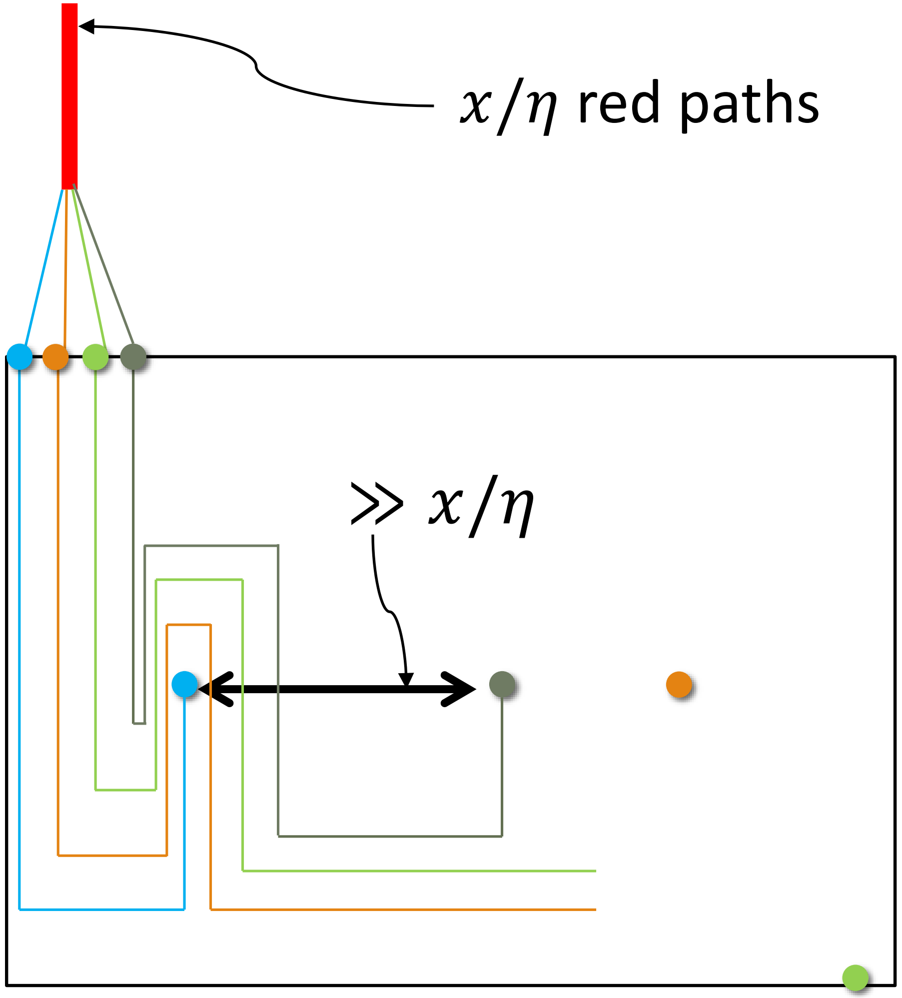

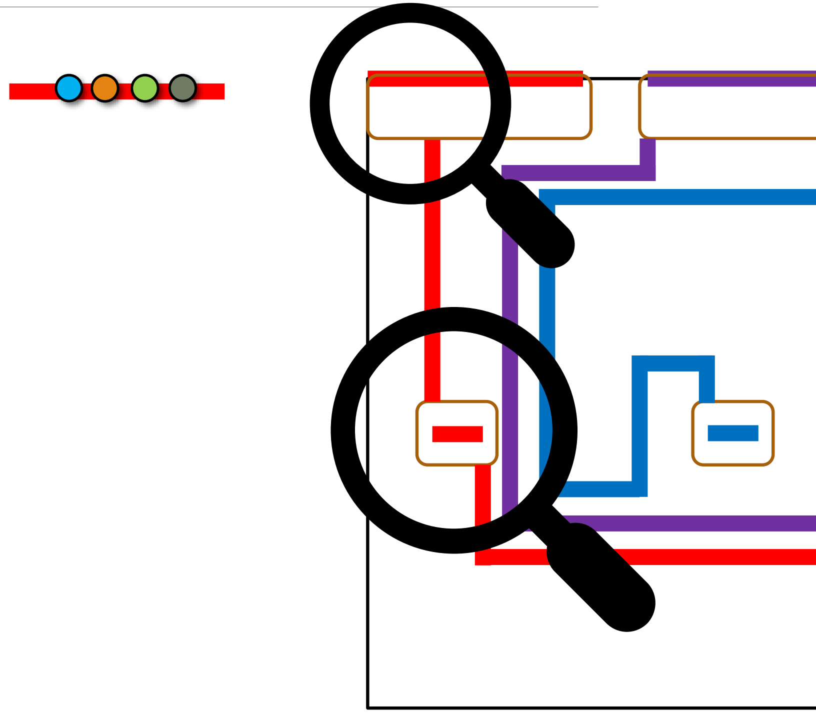

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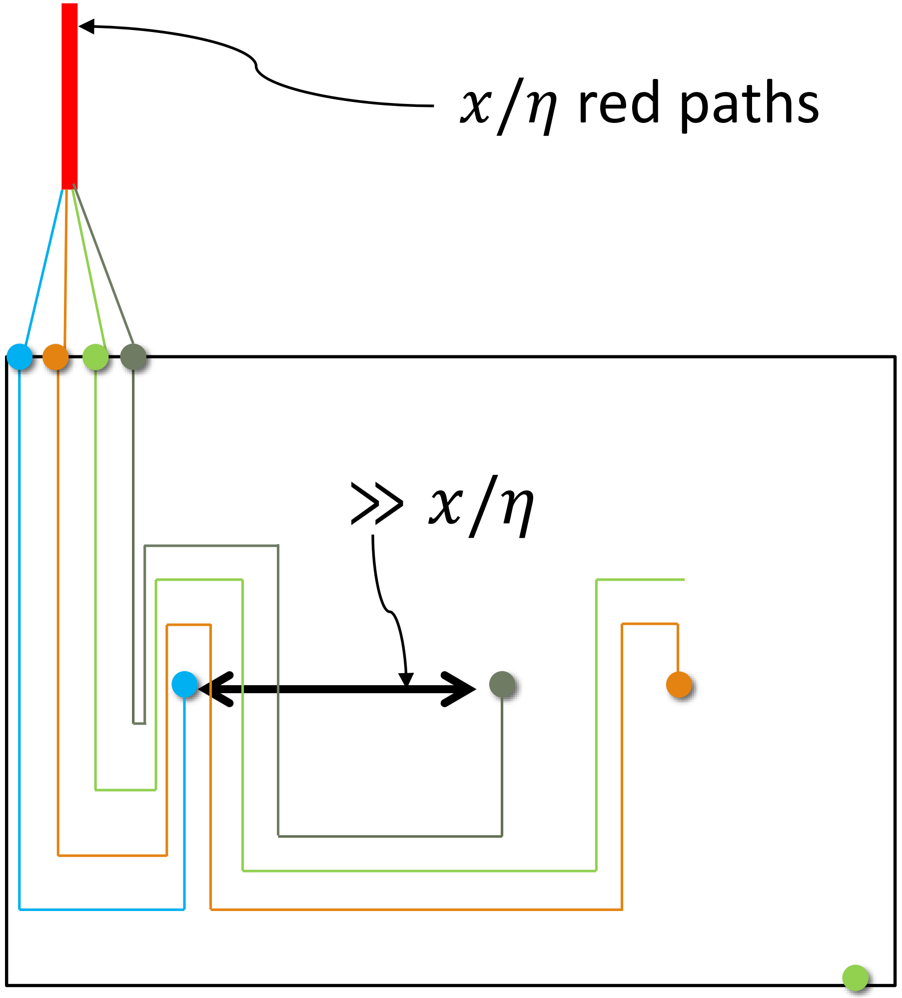

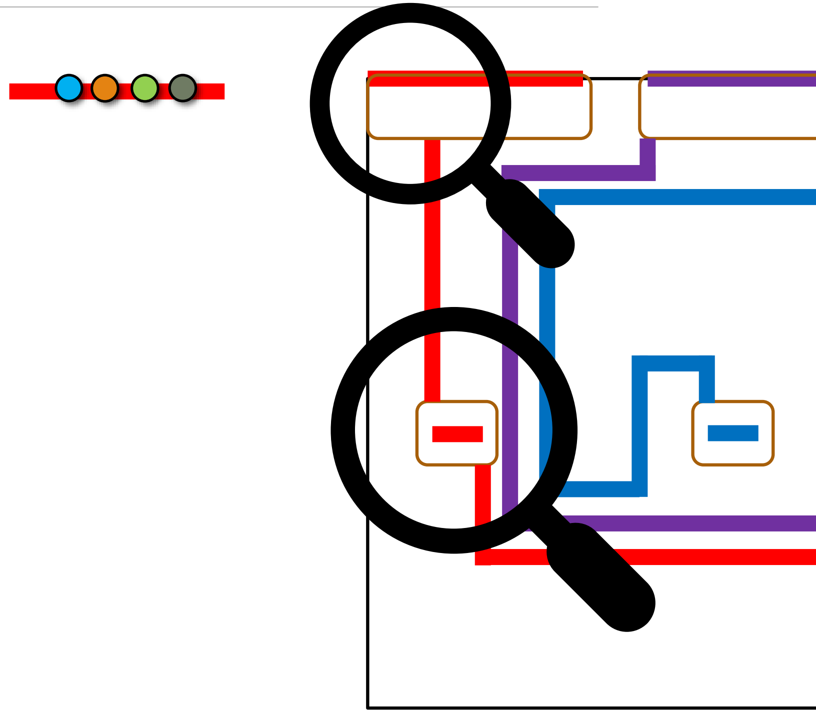

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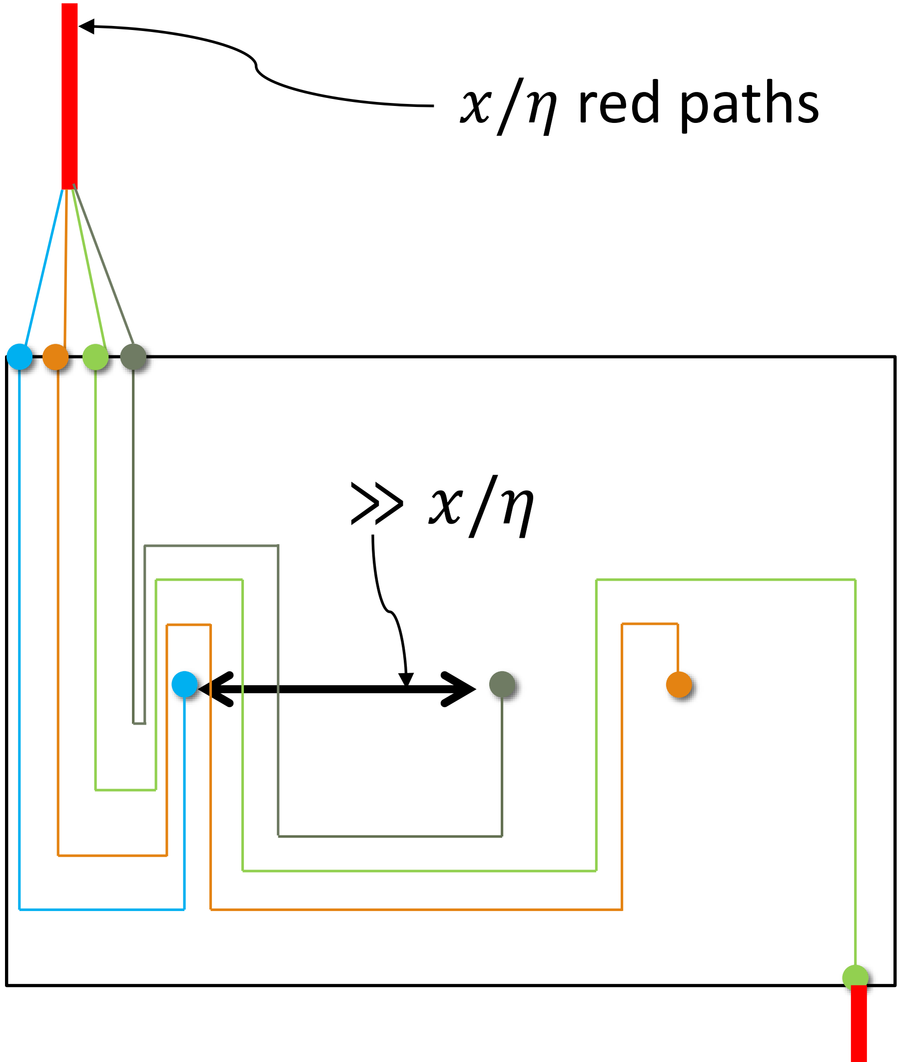

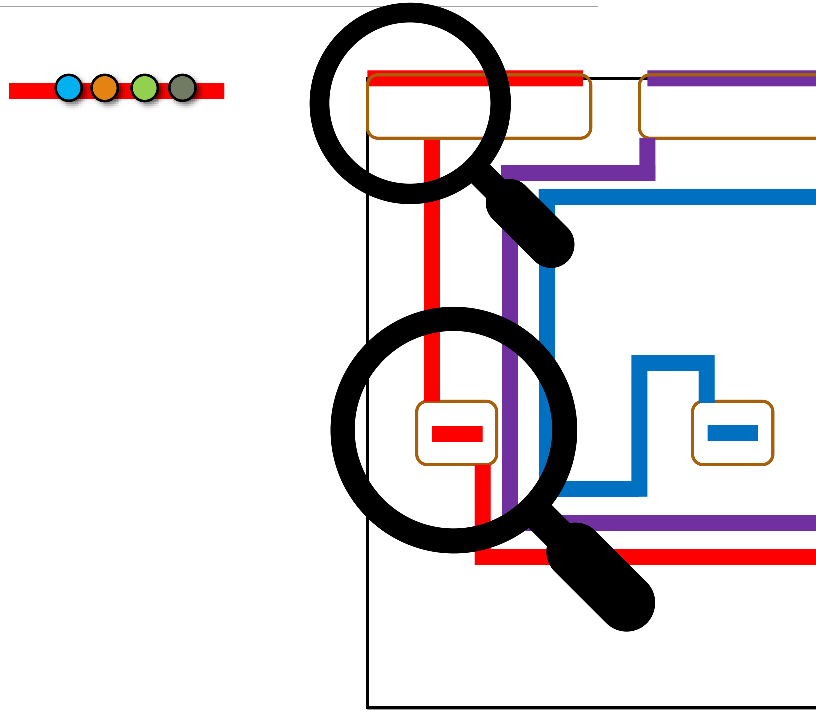

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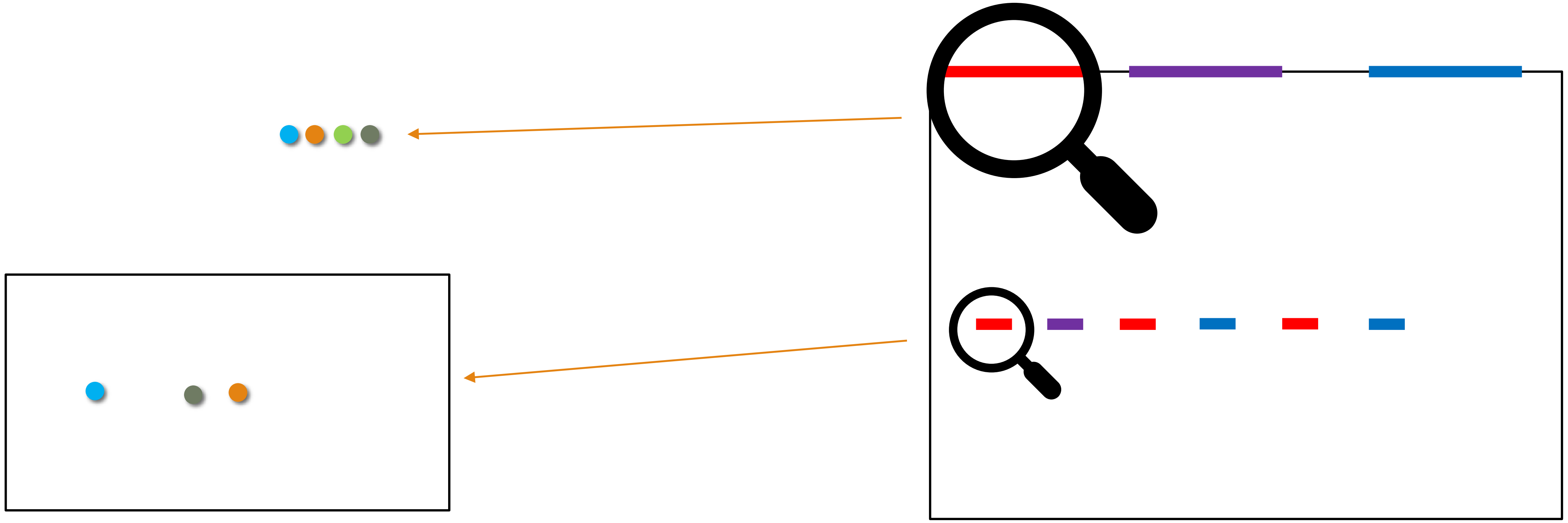
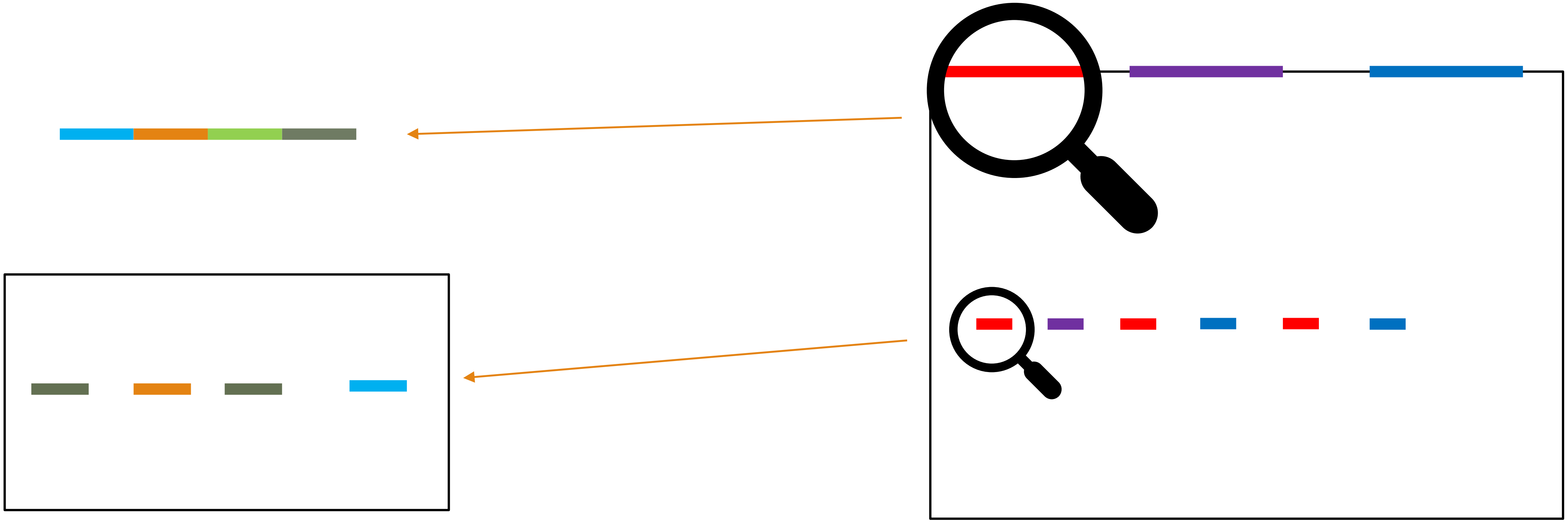
Recursive Partitioning



Recursive Partitioning

To optimize our approximation ratio, we set $\eta = 2^{\sqrt{\log n}}$ and extend this approach to $\sqrt{\log n}$ levels



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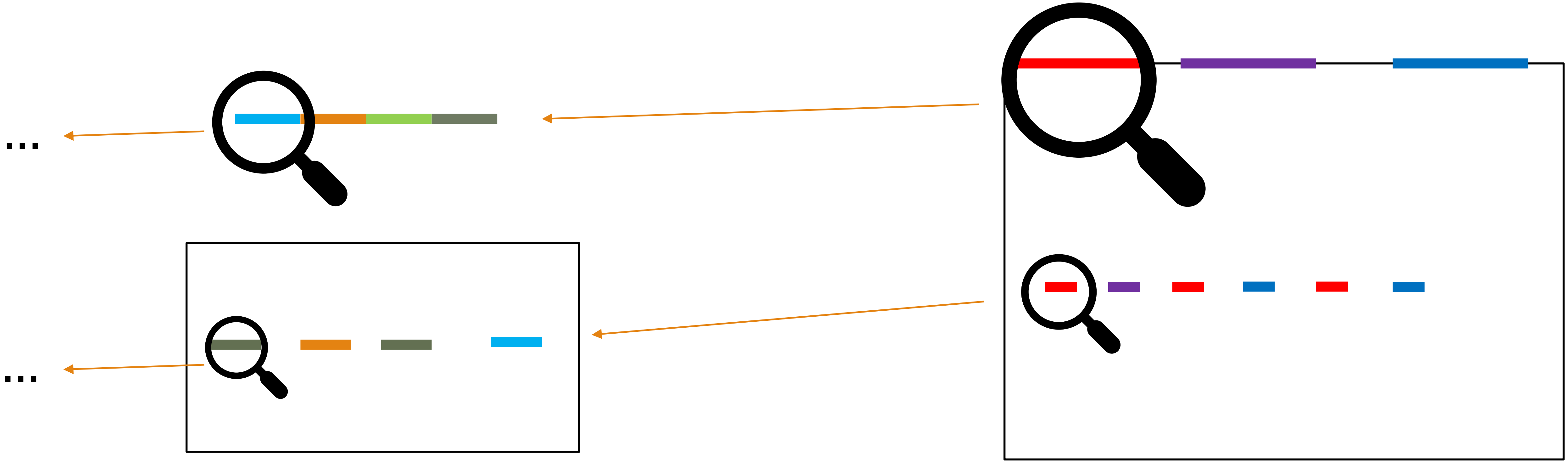
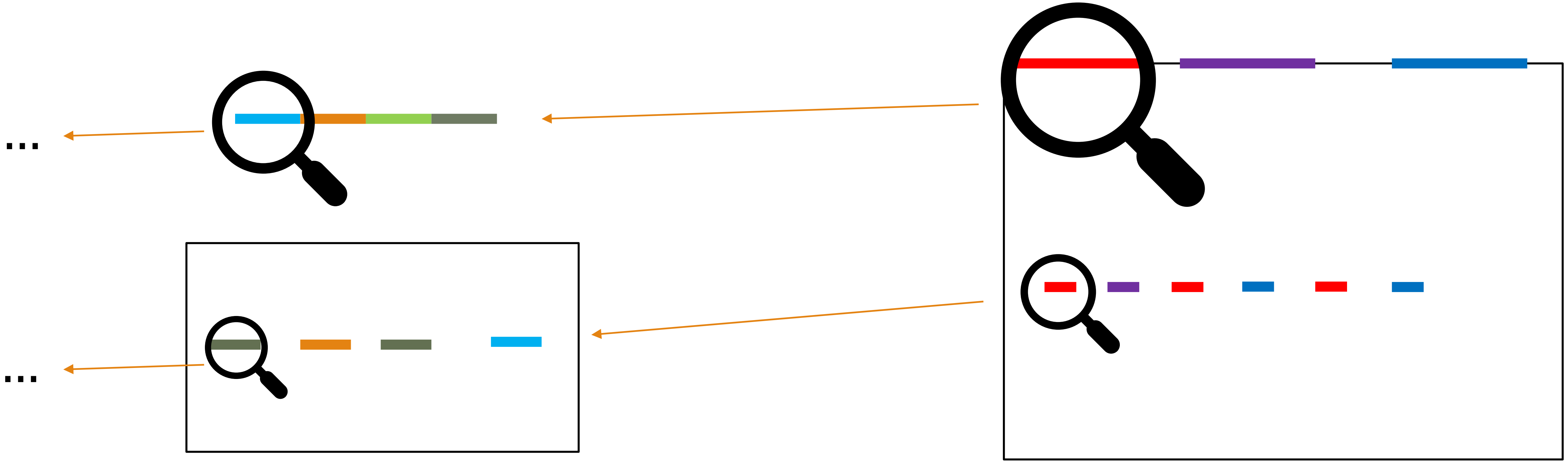


Recursive Partitioning

Recursive Partitioning = “*hierarchical set of intervals*” and “*hierarchical assignment*” of colors



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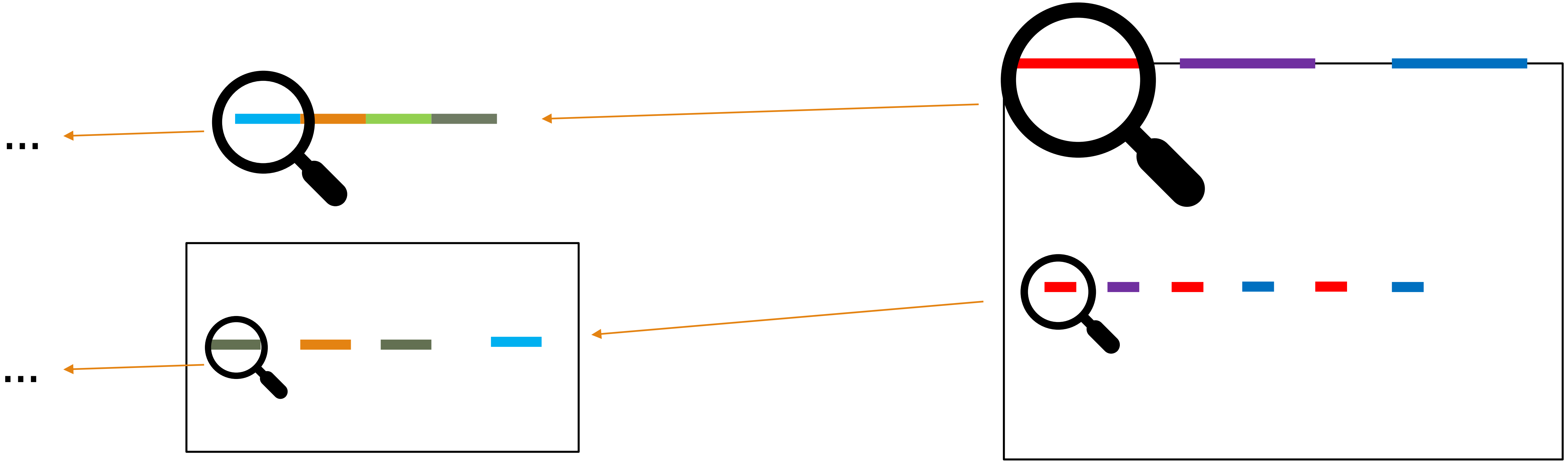
Nested intervals on source-row and nested intervals on destination-row



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Each interval on destination-row is hierarchically mapped to a unique interval on source-row

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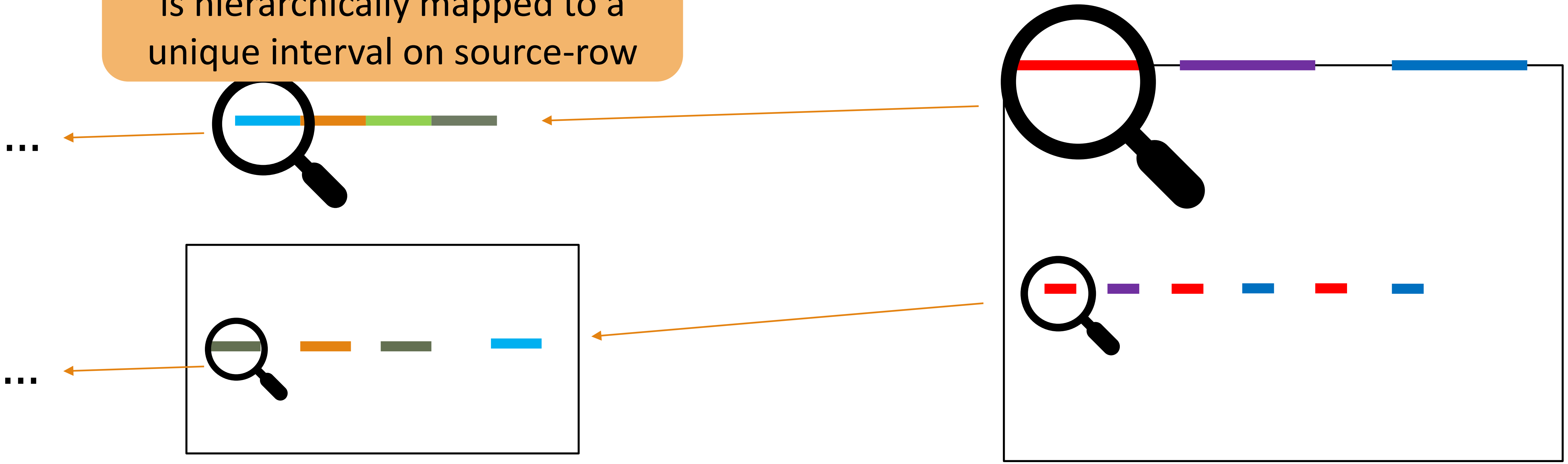

Recursive Partitioning | Why?

Theorem: $OPT_{RP} \geq OPT/2^{\tilde{O}(\sqrt{\log n})}$

Largest subset of demand pairs with Recursive Partitioning Property

Value of the optimum NDP-Grid solution

Recursive Partitioning | How?

Theorem: Can efficiently find a set of $OPT_{RP}/2^{O(\sqrt{\log n})}$ demand pairs with Recursive Partitioning Property

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Idea:

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- Solve for each candidate separately
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Write LP and perform randomized rounding level by level

Final Algorithm

- Find a set of OPT_{RP} / $2^{O\left(\sqrt{\log n}\right)}$ demand pairs with Recursive Partitioning Property
- Route all of them!

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Recall: $OPT_{RP} \geq OPT/2^{\tilde{O}\left(\sqrt{\log n}\right)}$

We route $OPT/2^{\tilde{O}\left(\sqrt{\log n}\right)}$ demand pairs

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- $2^{\tilde{O}(\sqrt{\log n})}$-approximation algorithm for NDP-Grids if sources appear on the boundary

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  Better hardness results for this case?
- Congestion minimization?
  - Route everything, but minimize *load*
  - $O(\log n / \log\log n)$-approx
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Thank You!