Regarding Counterexample of “Convergence in probability implies convergence almost everywhere”

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January 16, 2013

A variant of Type-writer sequence\(^1\) was presented in class as a counterexample of the converse of the statement “Almost everywhere convergence implies convergence in probability.”

The random variables were defined as follows: Let \( \{a_n\}_{n>0} \) be a sequence defined as \( a_n = (\sum_{i=1}^n \frac{1}{i})(mod1) \) for all \( n \). Now \( Y_n \) was defined for the measure space \((0,1], B, \mu)\) where \( B \) was the Borel sigma algebra and \( \mu \) was the uniform probability. \( Y_n \) was defined as follows:

\[
Y_n(\omega) = \begin{cases} 
1 & \text{if } 0 \leq a_n < a_{n+1} \leq 1 \text{ and } \omega \in [a_n, a_{n+1}] \\
1 & \text{if } 0 \leq a_{n+1} < a_n \leq 1 \text{ and } \omega \in [0, a_{n+1}] \cup [a_n, 1] \\
0 & \text{otherwise}
\end{cases}
\]

It was proved in class that \( \{Y_n\}_{n>0} \) be a sequence of random variable converging to a random variable \( Y \) in probability but not almost everywhere.

Here we present reasons why a substantially different counterexample can’t be constructed. Let \( \{X_n\}_{n>0} \) is a sequence of random variable which converges to a random variable \( X \) in measure, but not almost everywhere.

**Claim 1.** If for all \( \varepsilon > 0 \),

\[
\sum_{n=1}^{\infty} P\{|X_n - X| > \varepsilon\} < \infty
\]

then \( X_n \xrightarrow{a.e.} X \).

**Proof.** We want to show the set \( B \), on which \( X_n \) does not converge to \( X \) point-wise has measure 0.

For the sake of contradiction, let us assume that \( X_n \) does not converge point-wise to \( X \) on a set \( A \) of positive measure.

So there is an \( \varepsilon_1 > 0 \), so that the set:

\[
A(\varepsilon_1) = \{\omega : |X_n(\omega) - X(\omega)| > \varepsilon_1 \text{ for infinitely many } n\}
\]

\(1\) Type-writer sequence is defined in “http://terrytao.wordpress.com/2010/10/02/245a-notes-4-modes-of-convergence/”, Example 4
By triangular inequality, we know that for

\[ |X_n(\omega) - X(\omega)| > \epsilon_1 \]

Now, \( A(\epsilon_1) = \bigcup_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega : |X_n(\omega) - X(\omega)| > \epsilon_1\} \)

So, \( A(\epsilon_1) \subseteq \bigcup_{n=m}^{\infty} \{\omega : |X_n(\omega) - X(\omega)| > \epsilon_1\} \) where \( m \) is any finite integer.

For this \( \epsilon_1 \), the summation on the left side of the assumption converges to some finite value. So, for all \( \delta > 0 \) there is an \( n_\delta \) such that \( \sum_{n=n_\delta}^{\infty} P(|X_n - X| > \epsilon_1) < \delta \).

So, setting \( m \) to \( n_\delta \),

\[
P(A) \leq P(\bigcup_{n=n_\delta}^{\infty} \{\omega : |X_n(\omega) - X(\omega)| > \epsilon_1\})
\leq \sum_{n=n_\delta}^{\infty} P(|X_n - X| > \epsilon_1)
< \delta
\]

This holds for all \( \delta > 0 \). So, \( P(A(\epsilon_1)) = 0 \) – a contradiction.

So, \( X_n \overset{a.e.}\to X \).

If \( X_n \) converges to \( X \) in probability, we have to have \( \forall \epsilon > 0, P(|X_n - X| > \epsilon) \to 0 \). But Claim 1 suggests that if such a sequence has to not converge point-wise, it is necessary that for some \( \epsilon > 0 \) the series \( P(|X_n - X| > \epsilon) \) diverges. In fact, if \( \epsilon_1 \) is such an \( \epsilon \), then for any \( 0 < \delta < \epsilon_1 \), \( P(|X_n - X| > \delta) \) is such a converging sequence whose series diverges. In the example given in class random variables were chosen in such a way that, \( \epsilon_1 = 1 \) and \( P(|X_n - X| > \delta) = \frac{1}{n} \)

Now suppose that \( X_n \) converges to \( X \) in probability, but does not converge point-wise on a positive measure set. So, there exists an \( r > 0 \), such that

\[
S(r) = \left\{ \omega : \lim\sup_n X_n(\omega) - \lim\inf_n X_n(\omega) > r \right\}
\]

has positive measure.

For a point \( \omega \in \Omega \) and an \( \epsilon > 0 \), define discrepancy index set \( I_{\omega,\epsilon} = \{n : |X_n(\omega) - X(\omega)| > \epsilon \} \).

**Claim 2.** For any point \( \omega_1 \in S(r) \), for almost all \( \omega_2 \in S(r) \), \( \text{card}(I_{\omega_1,r/2} \triangle I_{\omega_2,r/2}) = \infty \)

**Proof.** By triangular inequality, we know that for \( m > 0 \),

\[
S(m) \subseteq \left\{ \omega : \lim\sup_n X_n(\omega) - X(\omega) > m/2 \right\} \bigcap \left\{ \omega : \lim\inf_n X_n(\omega) - X(\omega) > m/2 \right\}
\]

In particular, \( 0 < P(S(r)) < P\left\{ \omega : \lim\sup_n X_n(\omega) - X(\omega) > r/2 \right\} \).

But, since \( \{X_n\} \) converges in probability to \( X \), \( P(|X_n - X| > r/2) \to 0 \) as \( n \to \infty \). Fix an \( \epsilon_1 > 0 \). There exists an \( n_{\epsilon_1} \) such that for all \( n > n_{\epsilon_1} \), \( P(|X_n - X| > r/2) < \epsilon_1 \).

Suppose \( I_{\omega_2,r/2} \) differs from \( I_{\omega_1,r/2} \) only in finitely many places for two distinct elements \( \omega_1 \) and \( \omega_2 \) of \( S(r) \). Let \( T(\omega_1) \) be set of all such \( \omega_2 \). So, for every
large enough \( n > n_1 \) (say), \(|X_n(\omega_1) - X(\omega_1)| > r/2 \iff |X_n(\omega_2) - X(\omega_2)| > r/2\).

Clearly, \( T(\omega_1) = \bigcup_{m=1}^{\infty} \bigcap_{n \geq m, \omega \in I_{\omega_1, r/2}} \{ \omega : |X_n(\omega) - X(\omega)| > r/2 \} \).

Let

\[ M_m(\omega_1) = \bigcap_{n \geq m, \omega \in I_{\omega_1, r/2}} \{ \omega : |X_n(\omega) - X(\omega)| > r/2 \} \]

Now for a fixed \( \omega_1 \) and for \( m > \max\{n_{\varepsilon_1}, n_1\} \),

\[ P(M_m(\omega_1)) \leq P(\omega : |X_{m_1}(\omega) - X(\omega)| > r/2) < \varepsilon_1 \]

where \( m_1 \in I_{\omega_1, r/2} \) and \( m_1 > m \). Now \( M_m(\omega_1) \uparrow T(\omega_1) \). So, \( P(T(\omega_1)) < \varepsilon_1 \).

Since \( \varepsilon_1 \) and \( \omega_1 \) were arbitrary, we have the proof. \( \square \)

Claim 2 suggests that for any two different points in the subset where \( X_n \) does not converge point-wise, the “cause(s)” of divergence \((I_{\omega, \varepsilon})\) are infinitely often different. So, in some way, discrepancy creating random variables have to be wrapped around that set infinitely often.

These two seem to have been the key ideas in the counter-example presented in the class. It seems that both these ideas are unavoidable: any such counter-example has to have these properties.