Mathematical Toolkit **Autumn 2025**

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1 Existence of eigenvalues

We now prove that a self-adjoint operator must have at least one eigenvalue, which is something we assumed in the proof the real spectral theorem in the previous lecture. Let us assume for now that V is an inner product space over $\mathbb{C}$. In this case we don't need self-adjointness to guarantee an eigenvalue. We thus prove the following more general result

Proposition 1.1 *Let V be a finite dimensional vector space over $\mathbb{C}$ and let $\varphi : V \to V$ be a linear operator. Then φ has at least one eigenvalue.*

Proof: Let $\dim(V) = n$. Let $v \in V \setminus 0_V$ be any non-zero vector. Consider the set of $n+1$ vectors $\{v, \varphi(v), \ldots, \varphi^n(v)\}$. Since the dimension of V is n, there must exists $c_0, \ldots, c_n \in \mathbb{C}$ such that

$$c_0 \cdot v + c_1 \cdot \varphi(v) + \cdots + c_n \varphi^n(v) \;=\; 0_V \,.$$

We assume above that $c_n \neq 0$, otherwise we can only consider the sum to the largest i such that $c_i \neq 0$. Let $P(x)$ denote the polynomial $c_0 + c_1 x + \cdots + c_n x^n$. Then the above can be written as $(P(\varphi))(v) = 0$, where $P(\varphi) : V \to V$ is a linear operator defined as

$$P(\varphi) \;:=\; c_0 \cdot \mathsf{id} + c_1 \cdot \varphi + \cdots + c_n \varphi^n \,,$$

with id used to denote the identity operator. Since P is a degree-n polynomial over $\mathbb{C}$, it can be factored into n linear factors, and we can write $P(x) = c_n \prod_{i=1}^{n} (x - \lambda_i)$ for $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. This means that we can write

$$P(\varphi) \;=\; c_n(\varphi - \lambda_n \cdot \mathsf{id}) \cdots (\varphi - \lambda_1 \cdot \mathsf{id}) \,.$$

Let $w_0 = v$ and define $w_i = \varphi(w_{i-1}) - \lambda_i \cdot w_{i-1}$ for $i \in [n]$. Note that $w_0 = v \neq 0_V$ and $w_n = P(\varphi)(v) = 0_V$. Let i^* denote the largest index i such that $w_i \neq 0_V$. Then, we have

$$0_V \;=\; w_{i^*+1} \;=\; \varphi(w_{i^*}) - \lambda_{i^*+1} \cdot w_{i^*} \,.$$

This implies that w_{i^*} is an eigenvector with eigenvalue λ_{i^*+1}. ■

To prove existence of eigenvalues for self-adjoint operators (acting on a vector space, which may not necessarily be over the field $\mathbb{C}$) using this, we note that $\varphi = \varphi^*$ implies the eigenvalue found by the above proposition must are real.

Exercise 1.2 *Use the fact that the eigenvalues of a self-adjoint operator are real to prove existence of eigenvalues even when V is an inner product space over $\mathbb{R}$.*

Hint: Define a "complex extension" $V' = \{u + iv \mid u, v \in V\}$, which is a vector space over $\mathbb{C}$ with the scalar multiplication rule

$$(a + ib) \cdot (u + iv) \;=\; (a \cdot u - b \cdot v) + i(a \cdot v + b \cdot u)\,.$$

Also, extend φ to φ' defined as $\varphi' : V' \to V'$ with $\varphi'(u + iv) = \varphi(u) + i\varphi(v)$. Then, φ' has at least one (possibly complex) eigenvalue by the previous result. Can you use it to deduce the existence of a real eigenvalue for φ.

2 Rayleigh quotients: eigenvalues as optimization

Definition 2.1 *Let $\varphi : V \to V$ be a self-adjoint linear operator and $v \in V \setminus \{0_V\}$. The* Rayleigh quotient *of φ at v is defined as*

$$\mathcal{R}_\varphi(v) \;:=\; \frac{\langle v, \varphi(v)\rangle}{\|v\|^2}\,.$$

Proposition 2.2 *Let $\dim(V) = n$ and let $\varphi : V \to V$ be a self-adjoint operator with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then,*

$$\lambda_1 \;=\; \max_{v \in V \setminus \{0_V\}} \mathcal{R}_\varphi(v) \quad \textit{and} \quad \lambda_n \;=\; \min_{v \in V \setminus \{0_V\}} \mathcal{R}_\varphi(v)$$

Using the above, Rayleigh quotients can be used to prove the spectral theorem for Hilbert spaces, by showing that the above maximum[1] is attained at a point in the space, and defines an eigenvalue if the operator φ is "compact". A proof can be found in these notes by Paul Garrett [**?**].

[1] Strictly speaking, we should write sup and inf instead of max and min until we can justify that max and min are well defined. The difference is that sup and inf are defined as limits while max and min are defined as actual maximum and minimum values in a space, and these may not always exist while we are at looking infinitely many values. Thus, while $\sup_{x \in (0,1)} x = 1$, the quantity $\max_{x \in (0,1)} x$ does not exist. However, in the cases we consider, the max and min will always exist (since our spaces are closed under limits) and we will use max and min in the class to simplify things.

Proposition 2.3 (Courant-Fischer theorem) *Let* $\dim(V) = n$ *and let* $\varphi : V \to V$ *be a self-adjoint operator with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$*. Then,*

$$\lambda_k = \max_{\substack{S \subseteq V \\ \dim(S)=k}} \min_{v \in S \setminus \{0_V\}} \mathcal{R}_\varphi(v) = \min_{\substack{S \subseteq V \\ \dim(S)=n-k+1}} \max_{v \in S \setminus \{0_V\}} \mathcal{R}_\varphi(v) .$$

Definition 2.4 *Let* $\varphi : V \to V$ *be a self-adjoint operator.* Φ *is said to be* positive semidefinite *if* $\mathcal{R}_\varphi(v) \geq 0$ *for all* $v \neq 0$*.* Φ *is said to be* positive definite *if* $\mathcal{R}_\varphi(v) > 0$ *for all* $v \neq 0$*.*

Proposition 2.5 *Let* $\varphi : V \to V$ *be a self-adjoint linear operator. Then the following are equivalent:*

1. $\mathcal{R}_\varphi(v) \geq 0$ *for all* $v \neq 0$*.*
2. *All eigenvalues of* φ *are non-negative.*
3. *There exists* $\alpha : V \to V$ *such that* $\varphi = \alpha^*\alpha$*.*

The decomposition of a positive semidefinite operator in the form $\varphi = \alpha^*\alpha$ is known as the Cholesky decomposition of the operator. Note that if we can write φ as $\alpha^*\alpha$ for any $\alpha : V \to W$, then this in fact also shows that φ is self-adjoint and positive semidefinite.