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1 Fourier coefficients

Let V be a finite dimensional inner product space and let $B = \{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then for any $v \in V$, there exist $c_1, \ldots, c_n \in \mathbb{F}$ such that $v = \sum_i c_i \cdot w_i$. The coefficients c_i are often called Fourier coefficients (of v, with respect to the basis B). Using the orthonormality and the properties of the inner product, we get

Proposition 1.1 *Let $B = \{w_1, \ldots, w_n\}$ be an orthonormal basis for V, and let $v \in V$ be expressible as $v = \sum_{i=1}^n c_i \cdot w_i$. Then, for all $i \in [n]$, we must have $c_i = \langle w_i, v\rangle$.*

This can be used to prove the following

Proposition 1.2 (Parseval's identity) *Let V be a finite dimensional inner product space and let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then, for any $u, v \in V$*

$$\langle u, v\rangle \;=\; \sum_{i=1}^{n} \overline{\langle w_i, u\rangle} \cdot \langle w_i, v\rangle \; .$$

Exercise 1.3 *Prove that the set of functions*

$$S \;=\; \{1/2\} \cup \{\sin(k\pi x) \mid k \in \mathbb{N}, k \geq 1\} \cup \{\cos(k\pi x) \mid k \in \mathbb{N}, k \geq 1\} \; ,$$

is an orthonormal set in the Hilbert space of continuous real-valued functions mapping $[-1,1]$ to $\mathbb{R}$ (denoted $C([-1,1],\mathbb{R})$) under the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$.

In fact, the above functions form an orthonormal (Hilbert) basis for the space $C([-1,1],\mathbb{R})$, and are often referred to as the Fourier basis functions in signal analysis.

Parseval's identity can also be used to show that the size of the set large Fourier coefficients is small, and this is often very useful in "truncating" a signal to its large Fourier coefficients.

Exercise 1.4 *Let $B = \{w_1, \ldots, w_n\}$ be an orthonormal basis for a Hilbert space V. Let $v \in V$ with $\|v\| = 1$ be expressed as $v = \sum_i c_i \cdot w_i$. Then show that for any $\delta > 0$*

$$|\{i \;\mid\; |c_i| \geq \delta\}| \;\leq\; \frac{1}{\delta^2} \, .$$

2 Adjoints

Definition 2.1 *Let V, W be inner product spaces over the same field $\mathbb{F}$ and let $\varphi : V \to W$ be a linear transformation. A transformation $\varphi^* : W \to V$ is called an* adjoint *of φ if*

$$\langle w, \varphi(v) \rangle \;=\; \langle \varphi^*(w), v \rangle \quad \forall v \in V, w \in W\,.$$

Example 2.2 *Let $V = W = \mathbb{C}^n$ with the inner product $\langle u, v \rangle = \sum_{i=1}^{n} u_i \cdot \overline{v_i}$. Let $\varphi : V \to V$ be represented by the matrix A. Then φ^* is represented by the matrix $\overline{A^T}$.*

Example 2.3 *Let $V = C([0,1],[-1,1])$ with the inner product defined as $\langle f_1, f_2 \rangle = \int_0^1 f_1(x) f_2(x) dx$, and let $W = C([0,1/2],[-1,1])$ with the inner product $\langle g_1, g_2 \rangle = \int_0^{1/2} g_1(x) g_2(x) dx$. Let $\varphi : V \to W$ be defined as $\varphi(f)(x) = f(2x)$. Then, $\varphi^* : W \to V$ can be defined as*

$$\varphi^*(g)(y) = (1/2) \cdot g(y/2)\,.$$

Exercise 2.4 *Let $\varphi_{\text{left}} : Fib \to Fib$ be the left shift operator as before, and let $\langle f, g \rangle$ for $f, g \in Fib$ be defined as $\langle f, g \rangle = \sum_{n=0}^{\infty} \frac{f(n)g(n)}{C^n}$ for $C > 4$. Find φ^*_{left}.*

We will prove that every linear transformation has a unique adjoint. However, we first need the following characterization of linear transformations from V to $\mathbb{F}$.

Proposition 2.5 (Riesz Representation Theorem) *Let V be a finite-dimensional inner product space over $\mathbb{F}$ and let $\alpha : V \to \mathbb{F}$ be a linear transformation. Then there exists a unique $z \in V$ such that $\alpha(v) = \langle z, v \rangle \;\forall v \in V$.*

We only prove the theorem here for finite-dimensional spaces. However, the theorem holds for any Hilbert space, as long as the linear transformation is "continuous"'.

Proof: Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then check that

$$z \;=\; \sum_{i=1}^{n} \overline{\alpha(w_i)} \cdot w_i$$

must be the unique z satisfying the required property. ■

This can be used to prove the following:

Proposition 2.6 *Let V, W be finite dimensional inner product spaces and let $\varphi : V \to W$ be a linear transformation. Then there exists a unique $\varphi^* : W \to V$, such that*

$$\langle w, \varphi(v) \rangle \;=\; \langle \varphi^*(w), v \rangle \quad \forall v \in V, w \in W\,.$$

Proof: For each $w \in W$, the map $\langle w, \varphi(\cdot)\rangle : V \to \mathbb{F}$ is a linear transformation (check!) and hence there exists a unique $z_w \in V$ satisfying $\langle w, \varphi(v)\rangle = \langle z_w, v\rangle \;\; \forall v \in V$. Consider the map $\beta : W \to V$ defined as $\beta(w) = z_w$. By definition of β,

$$\langle w, \varphi(v)\rangle \;\; = \;\; \langle \beta(w), v\rangle \quad \forall v \in V, w \in W \,.$$

To check that α is linear, we note that $\forall v \in V,\ \forall w_1, w_2 \in W$,

$$\langle \beta(w_1 + w_2), v\rangle \;=\; \langle w_1 + w_2, \varphi(v)\rangle \;=\; \langle w_1, \varphi(v)\rangle + \langle w_2, \varphi(v)\rangle \;=\; \langle \beta(w_1), v\rangle + \langle \beta(w_2), v\rangle \;,$$

which implies $\beta(w_1 + w_2) = \beta(w_1) + \beta(w_2)$ (why?) $\beta(c \cdot w) = c \cdot \beta(w)$ follows similarly.

■

Note that the above proof only requires the Riesz representation theorem (to define z_w), and hence also works for Hilbert spaces (when φ is continuous).

2.1 Self-adjoint transformations

We first focus on a special class of linear transformations, where the transformation φ is its own adjoint. Note that this requires that φ maps an inner-product space V to itself. However, as we will see later, this special case turns out to be extremely useful. In fact, the theory developed for these special transformations, also yields interesting results for general linear transformations (which we will prove later).

Definition 2.7 *A linear transformation $\varphi : V \to V$ is called* self-adjoint *if $\varphi = \varphi^*$. Note that such a transformation necessarily needs to map v to itself, and is thus a linear operator.*

Example 2.8 *The transformation represented by matrix $A \in \mathbb{C}^{n\times n}$ is self-adjoint if $A = \overline{A^T}$. Such matrices are called Hermitian matrices.*

The proposition below shows that self-adjoint transformations have some particularly nice properties, which makes them very convenient to work with.

Proposition 2.9 *Let V be an inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then*

- *All eigenvalues of φ are real.*
- *If $\{w_1, \ldots, w_n\}$ are eigenvectors corresposnding to distinct eigenvalues then they are mutually orthogonal.*

Proof: The first property can be observed by noting that if $v \in V \setminus \{0_V\}$ is an eigenvector with eigenvalue λ, then

$$\lambda \cdot \langle v, v \rangle \;=\; \langle v, \lambda \cdot v \rangle \;=\; \langle v, \varphi(v) \rangle \;=\; \langle \varphi^*(v), v \rangle \;=\; \langle \varphi(v), v \rangle \;=\; \overline{\lambda} \cdot \langle v, v \rangle \;.$$

Since $\langle v, v \rangle \neq 0$, we must have $\lambda = \overline{\lambda}$ which implies that $\lambda \in \mathbb{R}$. For the second part, observe that if $i \neq j$, then we have

$$\lambda_j \cdot \langle w_i, w_j \rangle \;=\; \langle w_i, \varphi(w_j) \rangle \;=\; \langle \varphi^*(w_i), w_j \rangle \;=\; \langle \varphi(w_i), w_j \rangle \;=\; \overline{\lambda_i} \cdot \langle w_i, w_j \rangle \;.$$

Since eigenvalues are real, we get $(\lambda_i - \lambda_j) \cdot \langle w_i, w_j \rangle = 0$, which implies $\langle w_i, w_j \rangle = 0$ using $\lambda_i \neq \lambda_j$. ■

3 The Real Spectral Theorem

We can now prove the ”real spectral theorem” for self-adjoint operators $\varphi : V \to V$ (so named because the eigenvalues of a self-adjoint operator are real, not because other spectral theorems are fake!) We will show that any such operator is not only diagonalizable (has a basis of eigenvectors) but is in fact *orthogonally diagonalizable* i.e., has an *orthonormal* basis of eigenvectors. This gives a very convenient way of thinking about the action of such operators. In particular, let $\dim(V) = n$ and $\{w_1, \ldots, w_n\}$ form an orthonormal basis of eigenvectors for φ, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any vector v expressible in this basis as (say) $v = \sum_{i=1}^{n} c_i \cdot w_i$, we can think of the action of φ as

$$\varphi(v) \;=\; \varphi\left(\sum_{i=1}^{n} c_i \cdot v_i\right) \;=\; \sum_{i=1}^{n} c_i \cdot \lambda_i \cdot w_i \,.$$

Of course, we can also think of the action of φ in this way as long as $w_1, \ldots, w_n$ form a basis (not necessarily orthonormal). However, this view is particularly useful when they form an orthonormal basis. As we will later see, this also provides the “right” basis to think about many matrices, such as the adjacency matrices of graphs (where such decompositions are the subject of spectral graph theory). To prove the spectral theorem, We will need the following statement (which we’ll prove later).

Proposition 3.1 *Let V be a finite-dimensional inner product space (over $\mathbb{R}$ or $\mathbb{C}$) and let $\varphi : V \to V$ be a self-adjoint linear operator. Then φ has at least one eigenvalue.*

Using the above proposition, we will prove the spectral theorem below for finite dimensional vector spaces. The proof below can also be made to work for Hilbert spaces (using the axiom of choice). The above proposition, which gives the existence of an eigenvalue is often proved differently for finite and infinite-dimensional spaces, and the proof for infinite-dimensional Hilbert spaces requires additional conditions on the operator φ. We first prove the spectral theorem assuming the above proposition.

Proposition 3.2 (Real spectral theorem) *Let V be a finite-dimensional inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then φ is orthogonally diagonalizable.*

Proof: By induction on the dimension of V. Let $\dim(V) = 1$. Then by the previous proposition φ has at least one eigenvalue, and hence at least one eigenvector, say w. Then $w/\|w\|$ is a unit vector which forms a basis for V.

Let $\dim(V) = k+1$. Again, by the previous proposition φ has at least one eigenvector, say w. Let $W = \text{Span}(\{w\})$ and let $W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0\}$. Check the following:

- $W^{\perp}$ is a subspace of V.
- $\dim(W^{\perp}) = k$.
- $W^{\perp}$ is invariant under φ i.e., $\forall v \in W^{\perp}$, $\varphi(v) \in W^{\perp}$.

Thus, we can consider the operator $\varphi' : W^{\perp} \to W^{\perp}$ defined as

$$\varphi'(v) \ := \ \varphi(v) \qquad \forall v \in W^{\perp}.$$

Then, φ' is a self-adjoint (check!) operator defined on the k-dimensional space $W^{\perp}$. By the induction hypothesis, there exists an orthonormal basis $\{w_1, \ldots, w_k\}$ for $W^{\perp}$ such that each w_i is an eigenvector of φ. Thus $\left\{w_1, \ldots, w_k, \frac{w}{\|w\|}\right\}$ is an orthonormal basis for V, comprising of eigenvectors of φ. ■