Mathematical Toolkit **Autumn 2023**

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1 Inner-products and Adjoints

Definition 1.1 *Let V, W be inner product spaces over the same field $\mathbb{F}$ and let $\varphi : V \to W$ be a linear transformation. A transformation $\varphi^* : W \to V$ is called an* adjoint *of φ if*

$$\langle w, \varphi(v) \rangle \;=\; \langle \varphi^*(w), v \rangle \quad \forall v \in V, w \in W\,.$$

Example 1.2 *Let $V = W = \mathbb{C}^n$ with the inner product $\langle u, v \rangle = \sum_{i=1}^{n} u_i \cdot \overline{v_i}$. Let $\varphi : V \to V$ be represented by the matrix A. Then φ^* is represented by the matrix $\overline{A^T}$.*

Example 1.3 *Let $V = C([0,1],[-1,1])$ with the inner product defined as $\langle f_1, f_2 \rangle = \int_0^1 f_1(x) f_2(x) dx$, and let $W = C([0,1/2],[-1,1])$ with the inner product $\langle g_1, g_2 \rangle = \int_0^{1/2} g_1(x) g_2(x) dx$. Let $\varphi : V \to W$ be defined as $\varphi(f)(x) = f(2x)$. Then, $\varphi^* : W \to V$ can be defined as*

$$\varphi^*(g)(y) = (1/2) \cdot g(y/2)\,.$$

Exercise 1.4 *Let $\varphi_{\text{left}} : Fib \to Fib$ be the left shift operator as before, and let $\langle f, g \rangle$ for $f, g \in Fib$ be defined as $\langle f, g \rangle = \sum_{n=0}^{\infty} \frac{f(n)g(n)}{C^n}$ for $C > 4$. Find φ_{left}^*.*

We will prove that every linear transformation has a unique adjoint. However, we first need the following characterization of linear transformations from V to $\mathbb{F}$.

Proposition 1.5 (Riesz Representation Theorem) *Let V be a finite-dimensional inner product space over $\mathbb{F}$ and let $\alpha : V \to \mathbb{F}$ be a linear transformation. Then there exists a unique $z \in V$ such that $\alpha(v) = \langle z, v \rangle\ \forall v \in V$.*

We only prove the theorem here for finite-dimensional spaces. However, the theorem holds for any Hilbert space, as long as the linear transformation is "continuous"'.

Proof: Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then check that

$$z \;=\; \sum_{i=1}^{n} \overline{\alpha(w_i)} \cdot w_i$$

must be the unique z satisfying the required property. ∎

This can be used to prove the following:

Proposition 1.6 *Let V, W be finite dimensional inner product spaces and let $\varphi : V \to W$ be a linear transformation. Then there exists a unique $\varphi^* : W \to V$, such that*

$$\langle w, \varphi(v) \rangle \;=\; \langle \varphi^*(w), v \rangle \quad \forall v \in V, w \in W\,.$$

Proof: For each $w \in W$, the map $\langle w, \varphi(\cdot) \rangle : V \to \mathbb{F}$ is a linear transformation (check!) and hence there exists a unique $z_w \in V$ satisfying $\langle w, \varphi(v) \rangle = \langle z_w, v \rangle \;\forall v \in V$. Consider the map $\beta : W \to V$ defined as $\beta(w) = z_w$. By definition of β,

$$\langle w, \varphi(v) \rangle \;=\; \langle \beta(w), v \rangle \quad \forall v \in V, w \in W\,.$$

To check that α is linear, we note that $\forall v \in V,\; \forall w_1, w_2 \in W$,

$$\langle \beta(w_1 + w_2), v \rangle \;=\; \langle w_1 + w_2, \varphi(v) \rangle \;=\; \langle w_1, \varphi(v) \rangle + \langle w_2, \varphi(v) \rangle \;=\; \langle \beta(w_1), v \rangle + \langle \beta(w_2), v \rangle\;,$$

which implies $\beta(w_1 + w_2) = \beta(w_1) + \beta(w_2)$ (why?) $\beta(c \cdot w) = c \cdot \beta(w)$ follows similarly.

∎

Note that the above proof only requires the Riesz representation theorem (to define z_w), and hence also works for Hilbert spaces (when φ is continuous).

2 Self-adjoint transformations

Definition 2.1 *A linear transformation $\varphi : V \to V$ is called* self-adjoint *if $\varphi = \varphi^*$. Note that such a transformation necessarily needs to map v to itself, and is thus a linear operator.*

Example 2.2 *The transformation represented by matrix $A \in \mathbb{C}^{n \times n}$ is self-adjoint if $A = \overline{A^T}$. Such matrices are called Hermitian matrices.*

Proposition 2.3 *Let V be an inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then*

- *All eigenvalues of φ are real.*

- *If $\{w_1, \ldots, w_n\}$ are eigenvectors corresposnding to distinct eigenvalues then they are mutually orthogonal.*

Proof: The first property can be observed by noting that if $v \in V \setminus \{0_V\}$ is an eigenvector with eigenvalue λ, then

$$\lambda \cdot \langle v, v \rangle \;=\; \langle v, \lambda \cdot v \rangle \;=\; \langle v, \varphi(v) \rangle \;=\; \langle \varphi^*(v), v \rangle \;=\; \langle \varphi(v), v \rangle \;=\; \overline{\lambda} \cdot \langle v, v \rangle \;.$$

Since $\langle v, v \rangle \neq 0$, we must have $\lambda = \overline{\lambda}$ which implies that $\lambda \in \mathbb{R}$. For the second part, observe that if $i \neq j$, then we have

$$\lambda_j \cdot \left\langle w_i, w_j \right\rangle \;=\; \left\langle w_i, \varphi(w_j) \right\rangle \;=\; \left\langle \varphi^*(w_i), w_j \right\rangle \;=\; \left\langle \varphi(w_i), w_j \right\rangle \;=\; \overline{\lambda_i} \cdot \left\langle w_i, w_j \right\rangle \;.$$

Since eigenvalues are real, we get $(\lambda_i - \lambda_j) \cdot \left\langle w_i, w_j \right\rangle = 0$, which implies $\left\langle w_i, w_j \right\rangle = 0$ using $\lambda_i \neq \lambda_j$. ■