1 Gaussian Random Variables

A Gaussian random variable $X$ is defined through the density function

$$\gamma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where $\mu$ is its mean and $\sigma^2$ is its variance, and we write $X \sim \mathcal{N}(\mu, \sigma^2)$. To see the definition gives a valid probability distribution, we need to show $\int_{-\infty}^{\infty} \gamma(x)\,dx = 1$. It suffices to show for the case that $\mu = 0$ and $\sigma^2 = 1$. First we show the integral is bounded.

Claim 1.1 $I = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$ is bounded.

Proof: We see that

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 2 \int_{0}^{\infty} e^{-x^2/2}\,dx \leq 2 \int_{0}^{2} 1\,dx + 2 \int_{2}^{\infty} e^{-x}\,dx = 4 + 2e^{-2},$$

where we use the fact that $I$ is even and after $x = 2$, $e^{-x^2/2}$ is upper bounded by $e^{-x}$.

Next we show that the normalization factor is $\sqrt{2\pi}$.

Claim 1.2 $I^2 = 2\pi$.

Proof:

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dxdy$$

$$= \int_{0}^{\infty} \int_{0}^{2\pi} e^{-r^2/2}rdrd\theta \quad (\text{let } x = r \cos \theta \text{ and } y = r \sin \theta)$$

$$= 2\pi \int_{0}^{\infty} e^{-s}\,ds \quad (\text{let } s = r^2/2)$$

$$= 2\pi.$$

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This completes the proof that the definition gives a valid probability distribution. Before proceeding to applications of Gaussian random variables, we prove the following fact which we will use repeatedly.

**Proposition 1.3** Let $Z = c_1 X_1 + c_2 X_2$, where $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$ are independent. Then $Z \sim \mathcal{N}(0, c_1^2 + c_2^2)$.

**Proof:** By a simple change of variable, we can check that the density function for $c_1 X_1$ is
\[
\frac{1}{\sqrt{2\pi|c_1|}} e^{-\frac{x^2}{2|c_1|}},
\]
which shows that $c_1 X_1 \sim \mathcal{N}(0, c_1^2)$, and similarly $c_2 X_2 \sim \mathcal{N}(0, c_2^2)$.

Next, we can check that if $X$ and $Y$ are independent random variables with densities $f$ and $g$, then for $Z = X + Y$, we have
\[
\mathbb{P}[Z \leq t] = \int_{-\infty}^{t} \left( \int_{-\infty}^{\infty} f(x) \cdot g(z-x) dx \right) dz,
\]
which gives the density of $Z$ as $h(z) = \int_{-\infty}^{\infty} f(x) \cdot g(z-x) dx$. Taking $X = c_1 X_1$ and $Y = c_2 X_2$, we get the density of $Z = c_1 X_1 + c_2 X_2$ is
\[
h(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi|c_1|}} e^{-\frac{x^2}{2|c_1|}} \cdot \frac{1}{\sqrt{2\pi|c_2|}} e^{-\frac{(z-x)^2}{2|c_2|}} dx.
\]

We leave it as an exercise to show that the above integral gives
\[
h(z) = \frac{1}{\sqrt{2\pi(c_1^2 + c_2^2)}} e^{-\frac{z^2}{2(c_1^2 + c_2^2)}},
\]
which implies $c_1 X_1 + c_2 X_2 \sim \mathcal{N}(0, c_1^2 + c_2^2)$.

One can obtain the following corollary using an inductive application of the above proposition.

**Corollary 1.4** Let $X_1, \ldots, X_n \sim \mathcal{N}(0, 1)$ be independent standard Gaussian random variables. Then, for any vector of coefficients $c = (c_1, \ldots, c_n)$, we have
\[
Z = c_1 X_1 + \cdots + c_n X_n \sim \mathcal{N}(0, \|c\|^2),
\]
where $\|c\|^2 = c_1^2 + \cdots + c_n^2$.

**Remark 1.5** Note that we need independence for general statements of the form “linear combination of Gaussians is a Gaussian”, and that the statement can fail when the Gaussians are not independent. For example, consider the random variables
\[
X_1 \sim \mathcal{N}(0, 1) \quad \text{and} \quad X_2 = \begin{cases} 
X_1 & \text{if } |X_1| \leq 1 \\
-X_1 & \text{if } |X_1| > 1
\end{cases}
\]
One can check that if we look at the variables $X_1$ and $X_2$, they are Gaussian random variables with mean 0 and variance 1. However, we have that

$$X_1 + X_2 = \begin{cases} 2X_1 & \text{if } |X_1| \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$ 

Thus, the linear combination is zero with positive probability, and is not a Gaussian distribution. When a collection of Gaussian random variables $X_1, \ldots, X_n$ satisfies that their linear combinations are also Gaussian, they are called “jointly Gaussian random variables. Thus, we proved that independent Gaussian random variables are also jointly Gaussian.

We prove a useful lemma for later use.

**Lemma 1.6** For $X \sim N(0,1)$ and $\lambda \in (0,1/2)$,

$$\mathbb{E}[e^{\lambda X^2}] = \frac{1}{\sqrt{1 - 2\lambda}}.$$ 

**Proof:**

$$\mathbb{E}[e^{\lambda X^2}] = \int_{-\infty}^{\infty} e^{\lambda x^2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(1-2\lambda) x^2/2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \sqrt{1-2\lambda} \quad \text{(let } y = \sqrt{1-2\lambda} x)$$

$$= \frac{1}{\sqrt{1-2\lambda}} \quad \blacksquare$$

Recall that a Gaussian random variable $X$ is defined through the density function

$$\gamma(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where $\mu$ is its mean and $\sigma^2$ is its variance, and we write $X \sim N(\mu, \sigma^2)$.