Mathematical Toolkit **Autumn 2023**

Homework 2

Due: October 26, 2023

Note: *You may discuss these problems in groups. However, you must write up your own solutions and mention the names of the people in your group. Also, please do mention any books, papers or other sources you refer to. It is recommended that you typeset your solutions in LaTeX.*

1. **First things first**. **[3+2+2+4]**

   The following are some useful results we have sketched in class and also used some of them in proofs. Make sure you know how to prove them! For all the parts below, let V be a finite dimensional inner-product space with dimension n.

   (a) Let $\{w_1, \ldots, w_k\}$ be an orthonormal subset of V. Show that it can be completed to an orthonormal basis of V i.e., there exist vectors $\{w_{k+1}, \ldots, w_n\}$ such that the set $\{w_1, \ldots, w_n\}$ forms an orthonormal basis of V.

   (b) Let $\{w_1, \ldots, w_n\}$ and $\{u_1, \ldots, u_n\}$ be two different orthonormal bases of V. Prove that for all $v_1, v_2 \in V$

   $$\sum_{i=1}^{n} \overline{\langle w_i, v_1 \rangle} \cdot \langle w_i, v_2 \rangle \;=\; \sum_{i=1}^{n} \overline{\langle u_i, v_1 \rangle} \cdot \langle u_i, v_2 \rangle \;.$$

   Note that this implies that for all $v \in V$, $\sum_{i=1}^{n} |\langle w_i, v \rangle|^2 \;=\; \sum_{i=1}^{n} |\langle u_i, v \rangle|^2$.

   (c) If V and W are finite-dimensional inner product spaces, and $\varphi : V \to W$ is a linear transformation, show that if φ^* is the adjoint of φ, then φ is the adjoint of φ^* i.e., $(\varphi^*)^* = \varphi$.

   (d) We call $\varphi : V \to V$ a unitary operator if $\varphi\varphi^* = \varphi^*\varphi = \mathrm{id}$. Show that φ is a unitary operator *if and only if* for any orthonormal basis $\{w_1, \ldots, w_n\}$ of V, the set $\{\varphi(w_1), \ldots, \varphi(w_n)\}$ is also an orthonormal basis.

2. **Inner products from positive definite operators**. **[3+3]**

   Let V be an inner product space over $\mathbb{C}$ and let $\varphi : V \to V$ be a self-adjoint positive definite operator i.e., $\langle v, \varphi(v) \rangle > 0$ for all $v \in V \setminus \{0_V\}$. Let $\mu : V \times V \to \mathbb{C}$ be the function defined as $\mu(v, w) = \langle v, \varphi(w) \rangle$. Show that:

   (a) The function μ also defines an inner product on the vector space V.

(b) The operator φ is also self-adjoint for the inner product defined by the function μ.

3. **Eigenvalue interlacing**. **[2+3+3+3]**

Let α be a self-adjoint operator on an n-dimensional inner-product space V, and let $w_0 \in V \setminus 0$ be a non-zero vector with $\|w_0\| = 1$. Let $W \subseteq V$ denote the subspace defined as $W := \{v \in V \mid \langle w_0, v \rangle = 0\}$. Let $\beta : W \to V$ be defined as

$$\beta(w) \;=\; \alpha(w) - \langle w_0, \alpha(w) \rangle \cdot w_0 \,.$$

(a) Show that β is in fact an operator from W to W i.e., for all $w \in W$, we have $\beta(w) \in W$.

(b) Show that $\beta : W \to W$ as defined above is self-adjoint.

(c) Let $\lambda_1 \geq \cdots \geq \lambda_n$ denote the eigenvalues of α and let $\mu_1 \geq \cdots \geq \cdots \mu_{n-1}$ denote the eigenvalues of β (since $\dim(W) = n-1$). Show that $\lambda_1 \geq \mu_1$.

(d) Show that the eigenvalues of α and β are *interlacing* i.e.,

$$\lambda_i \;\geq\; \mu_i \;\geq\; \lambda_{i+1} \quad \forall i \in [n-1] \,.$$

4. **Low-rank approximation in Frobenius norm**. **[2+3+2+5]**

This problem complements a result we prove in the lecture, that the singular value decomposition of a matrix A gives the best low-rank approximation to A in the spectral norm. Here, we will show that the SVD also gives the best approximation in the Frobenius norm defined as $\|M\|_F^2 = \sum_{ij} \left|M_{ij}\right|^2$.

Let $A \in \mathbb{R}^{m \times n}$ be a matrix with the singular value decomposition

$$A \;=\; \sum_{j=1}^{r} \sigma_j \cdot w_j v_j^T \,,$$

with $\sigma_1 \geq \cdots \geq \sigma_r > 0$. For $k \leq r$, we will show that $A_k = \sum_{j=1}^{k} \sigma_j w_j v_j^T$ is also the best approximation of A in the Frobenius norm i.e., for any $B \in \mathbb{R}^{m \times n}$ of rank at most k we have

$$\|A - B\|_F \;\geq\; \|A - A_k\|_F \,.$$

(a) Given any $B \in \mathbb{R}^{m \times n}$ of rank $t \leq k$, let $b_i \in \mathbb{R}^n$ denote i^{th} row of B (written as a column vector). Let $S = \operatorname{Span}(\{b_1, \ldots, b_m\})$. What is the dimension of the space S?

(b) Let $a_i \in \mathbb{R}^n$ be a (column) vector such that the i^{th} row of A is a_i^T. Show that

$$\|A - B\|_F^2 \;=\; \sum_{i=1}^{m} \|a_i - b_i\|_2^2 \;\geq\; \sum_{i=1}^{m} (\mathrm{dist}(a_i, S))^2 \,.$$

(c) Let S_0 denote the subspace of dimension at most k which minimizes the quantity $\sum_{i=1}^{m} (\mathrm{dist}(a_i, V))^2$ over all subspaces $V \subseteq \mathbb{R}^n$ with $\dim(V) \leq k$. Express S_0 in terms of the singular vectors of the matrix A.

(d) Using the characterization for S_0 derived above, show that

$$\sum_{i=1}^{m} (\mathrm{dist}(a_i, S_0))^2 \;=\; \|A - A_k\|_F^2 \,.$$

This completes the proof since then

$$\|A - B\|_F^2 \;\geq\; \sum_{i=1}^{m} (\mathrm{dist}(a_i, S))^2 \;\geq\; \sum_{i=1}^{m} (\mathrm{dist}(a_i, S_0))^2 \;=\; \|A - A_k\|_F^2 \,.$$