Mathematical Toolkit **Autumn 2023**

Homework 1

Due: October 12, 2023

Note: *You may discuss these problems in groups. However, you must write up your own solutions and mention the names of the people in your group. Also, please do mention any books, papers or other sources you refer to. It is recommended that you typeset your solutions in LaTeX.*

1. **Field trip**. **[4+4+3]**

   Recall that for a prime p, $\mathbb{F} = \mathbb{Q}^2$ (set of pairs of rational numbers) is a field with the notions of addition and multiplication defined as

   $$(a, b) + (c, d) \ = \ (a + c, b + d) \quad \text{and} \quad (a, b) \cdot (c, d) \ = \ (ac + pbd, ad + bc)\,.$$

   (a) What are the additive and multiplicative identities? What is the multiplicative inverse of (a, b) for $(a, b) \neq 0_{\mathbb{F}}$?

   (b) Does everything in part (a) still go through if $p = 6$ (and hence, not prime)? How about $p = 4$?

   (c) Taking p to be a prime number again, the set

   $$S = \{a + b\sqrt{p} \mid a, b \in \mathbb{Q}\} \ \subseteq \ \mathbb{R}$$

   can be thought of as a vector space over the field $\mathbb{Q}$. What is its dimension? What is a basis for it?

2. **Linear equations**. **[2+3+2]**

   Let $A \in \mathbb{F}_2^{m \times n}$ be a matrix with entries in the field $\mathbb{F}_2$ and let $m < n$. Let all rows of A be linearly independent in the vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$.

   (a) What is the dimension of the space $\ker(A)$?

   (b) How many vectors $x \in \mathbb{F}_2^n$ satisfy the system of equations $Ax = 0$? (Note that here 0 denotes the zero vector in $\mathbb{F}_2^m$.)

   (c) Let $b \in \mathbb{F}_2^m$ be such that the system of equations $Ax = b$ has at least one solution, say x_0. Show that $\{x - x_0 \mid Ax = b\} = \ker(A)$. What is the total number of solutions to the system $Ax = b$?

For this problem you may use the fact that for a matrix $A \in \mathbb{F}^{m \times n}$ for any field $\mathbb{F}$, if $R \subseteq \mathbb{F}^n$ denotes the set of its rows and $C \subseteq \mathbb{F}^m$ denotes the set of its columns, then

$$\dim(\mathrm{Span}\,(R)) \;=\; \dim(\mathrm{Span}\,(C))\,.$$

The quantity $\dim(\mathrm{Span}\,(R))$ is called the row-rank of A and $\dim(\mathrm{Span}\,(C))$ is called the column-rank of A.

3. **The space above**. **[3+5]**

   Let V be a vector space over the field $\mathbb{F}$ (not necessarily $\mathbb{R}$ or $\mathbb{C}$) and let $f : V \to [0,1]$ be a function satisfying $f(c \cdot v + d \cdot w) \geq \min\{f(v), f(w)\}$ for all $c, d \in \mathbb{F}$ and all $v, w \in V$. Show that

   (a) $f(0_V) \;\geq\; f(v)$ for all $v \in V$.

   (b) For any $t \in [0, f(0_V)]$, the space $V_t = \{v \in V \mid f(v) \geq t\}$ is a subspace of V.

4. **Inner Products**. **[5]**

   Consider the set $\mathbb{R}[x]$ of polynomials in a single variable x with coefficients in $\mathbb{R}$, which is a vector space over the field $\mathbb{R}$. Define the function $\mu : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}$ as

   $$\mu(P, Q) \;=\; \mathrm{degree}(P \cdot Q) \quad \text{for all } P, Q \in \mathbb{R}[x]\,,$$

   where $P \cdot Q$ denotes the product of the two polynomials P and Q (which is another polynomial). Is the function μ an inner product? Justify your answer.

5. **Eigenvalues**. **[8]**

   Let V be a finite dimensional vector space over a field $\mathbb{F}$ and $\alpha, \beta : V \to V$ be linear operators. Show that for every $\lambda \in \mathbb{F}$ (including $0_{\mathbb{F}}$), λ is an eigenvalue of $\alpha\beta$ if and only if λ is an eigenvalue of $\beta\alpha$. Here, $\alpha\beta$ denotes the linear transformation $\alpha \circ \beta$ defined as $\alpha\beta(v) = \alpha(\beta(v))\ \forall v \in V$ (and $\beta\alpha$ is defined similarly).

6. **Projections**. **[2+4+5+3+2+2+3]**

   A linear operator $\varphi : V \to V$ is called a projection if $\varphi^2 = \varphi$ i.e., $\varphi^2(v) := \varphi(\varphi(v)) = \varphi(v)\ \forall v \in V$. For the parts below, let V be a (not necessarily finite dimensional) vector space over a field $\mathbb{F}$ and let $\varphi : V \to V$ be a projection.

   (a) Show that $\psi : V \to V$ defined as $\psi(v) = v - \varphi(v)$ is also a projection.

   (b) Show that $\ker(\varphi) = \mathrm{im}(\psi)$ and $\mathrm{im}(\varphi) = \ker(\psi)$.

   (c) Show that any $v \in V$ can be uniquely decomposed as $v = u + w$, with $u \in \mathrm{im}(\varphi)$ and $w \in \ker(\varphi)$. We say that those two subspaces are complementary.

(d) What are the possible eigenvalues λ of φ and the respective eigenspaces, i.e., $U_\lambda := \{v \mid \varphi(v) = \lambda v\}$?

(e) Deduce that φ (and thus ψ) is diagonalizable.

(f) Let $V = \mathbb{R}^2$, and define the maps $\varphi_1, \varphi_2 : V \to V$ as

$$\varphi_1(x,y) \;=\; \left(\frac{x+y}{2}, \frac{x+y}{2}\right) \quad \text{and} \quad \varphi_2(x,y) \;=\; \left(\frac{2x+y}{3}, \frac{2x+y}{3}\right).$$

Show that $\mathrm{im}(\varphi_1) = \mathrm{im}(\varphi_2)$. Are their kernels the same? Prove that both φ_1 and φ_2 are projections.

(g) Given an inner product on V, we call a projection $\varphi : V \to V$ an orthogonal projection if $\ker(\varphi) = \mathrm{im}(\varphi)^\perp$, where $W^\perp := \{v \in V \mid \langle v, w\rangle = 0 \; \forall w \in W\}$ i.e., these two subspaces are orthogonal complements.

Consider $V = \mathbb{R}^2$ with the usual inner product, and take φ_1, φ_2 as defined above. Which of φ_1 and φ_2 is orthogonal and which isn't? Define a different inner product on $\mathbb{R}^2$ which flips your answer?