Mathematical Toolkit **Autumn 2021**

Lecture 16: November 30, 2021

Lecturer: Madhur Tulsiani

1 Random variables over uncountably infinite probability spaces

To define a random variable, we need to define a σ-algebra on the range of the random variable. A random variables is then defined as a *measurable* function from the probability space to the range: functions where the pre-image of every subset in the range σ-algebra is an event in $\mathcal{F}$.

An important case is when the range is $[0,1]$ or $\mathbb{R}$. In this case we say that we have a *real-valued* random variable, and we use the Borel σ-algebra unless otherwise noted. For countable probability spaces, we wrote the expectation of a real-valued random variable as a sum. For uncountable spaces, the expectation is an integral with respect to the measure.

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) d\nu \,.$$

The definition of the integral with respect to a measure requires some amount of care, though we will not be able to discuss this in much detail. Let ν be any probability measure over the space $\mathbb{R}$ equipped with the Borel σ-algebra. Define the function F as

$$F(x) := \nu((-\infty, x]) \,,$$

which is well defined since the interval $(-\infty, x]$ is in the Borel σ-algebra. This can be used to define a random variable X such that $\mathbb{P}[X \leq x] = F(x)$. The function F is known as the distribution function or the cummulative density function of X.

When the function F has the form

$$F(x) = \int_{-\infty}^{x} f(z) dz \,,$$

then f is called the density function of the random variable X. In this case, one typically refers to X as a "continuous" random variable. To calculate the above expectation for continuous random variables, we can use usual (Lebesgue) integration:

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) dx \,.$$

(The notion of density can be extended to between any two measures, via the Radon-Nikodym theorem. In that context, the density f of a continuous random variable is referred to as the Radon-Nikodym derivative with respect to the Lebesgue measure. In the earlier example with the measure concentrated on the finite set T, the probability of each point is the Radon-Nikodym derivative with respect to the counting measure of T: $\nu_T = \sum_{t \in T} \delta_t$.)

2 Gaussian Random Variables

A Gaussian random variable X is defined through the density function

$$\gamma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where μ is its mean and σ^2 is its variance, and we write $X \sim \mathcal{N}(\mu, \sigma^2)$. To see the definition gives a valid probability distribution, we need to show $\int_{-\infty}^{\infty} \gamma(x)dx = 1$. It suffices to show for the case that $\mu = 0$ and $\sigma^2 = 1$. First we show the integral is bounded.

Claim 2.1 *$I = \int_{-\infty}^{\infty} e^{-x^2/2}dx$ is bounded.*

Proof: We see that

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}dx = 2\int_0^{\infty} e^{-x^2/2}dx \leq 2\int_0^2 1dx + 2\int_2^{\infty} e^{-x}dx = 4 + 2e^{-2},$$

where we use the fact that I is even and after $x = 2$, $e^{-x^2/2}$ is upper bounded by e^{-x}. ■

Next we show that the normalization factor is $\sqrt{2\pi}$.

Claim 2.2 *$I^2 = 2\pi$.*

Proof:

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2/2}dx \int_{-\infty}^{\infty} e^{-y^2/2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}dxdy \\
&= \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} r dr d\theta \quad (\text{let } x = r\cos\theta \text{ and } y = r\sin\theta) \\
&= 2\pi \int_0^{\infty} e^{-s}ds \quad (\text{let } s = r^2/2) \\
&= 2\pi\,.
\end{aligned}$$

■

This completes the proof that the definition gives a valid probability distribution. We prove a useful lemma for later use.

Lemma 2.3 *For $X \sim \mathcal{N}(0,1)$ and $\lambda \in (0,1/2)$,*

$$\mathbb{E}\left[e^{\lambda \cdot X^2}\right] = \frac{1}{\sqrt{1-2\lambda}}.$$

Proof:

$$\begin{aligned}
\mathbb{E}\left[e^{\lambda \cdot X^2}\right] &= \int_{-\infty}^{\infty} e^{\lambda \cdot x^2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(1-2\lambda)x^2/2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \frac{dy}{\sqrt{1-2\lambda}} \quad (\text{let } y = \sqrt{1-2\lambda}x) \\
&= \frac{1}{\sqrt{1-2\lambda}}
\end{aligned}$$

■