Problems for Discussion 7

November 13, 2019

Exercise 1.
For a permutation σ on $[n]$, i is a fixed point if $\sigma(i) = i$. A cycle in the permutation σ is a set $(x_1, x_2, \cdots, x_k)$ such that $\sigma(x_i) = x_{i+1}$ for $1 \leq i \leq k-1$ and $\sigma(x_k) = x_1$ for some ordering of the set elements, and we say the length of this cycle is k. Under this notation, a fixed point is a cycle of length 1.

1. If a permutation is sampled uniformly at random from all permutations on $[n]$, find the expected number of fixed points.
2. If a permutation is sampled uniformly at random from all permutations on $[n]$, find the expected number of k-length cycles.

Exercise 2.
You have been given a fair and unbiased coin, that is, it gives head and tails with probability 0.5 each, and different tosses are independent.

1. Use this coin to generate an event that occurs with probability 0.3125.
2. Now generalize the idea to generate an event that occurs with probability 1/3.

For both parts above, your strategy should not use more than 2 coin tosses on average.

Hints/Solutions

Hint 1.

1. Let X_i be the indicator random variable of whether i is a fixed point, and use linearity of expectation along with $\Pr[X_i = 1] = 1/i$ to show there is one fixed point on average.

2. Generalize the above idea, by having an indicator random variable for every k-sized subset of $[n]$, and showing that the probability of every such subset being a k-cycle is $\frac{(k-1)!(n-k)!}{n!}$. This will give a total expected number of k-cycles as $1/k$.

Hint 2.

1. Toss the coin 4 times and say the event $E = \{HHHH, HHHT, HHTH, HHTT, HTHH\}$. Then, $\Pr[E] = 5/16 = 0.3125$. Note that if we get a T in the first toss, we can already stop knowing that E will not occur. This means we do not always need to toss all 4 coins. Similarly, we can stop if we got an H in first toss and H in the second toss as well, because then E is definitely going to occur.

   In fact, verify that expected number of coin tosses will be $1 \times \frac{8}{16} + 2 \times \frac{8}{16}\frac{4}{8} + 3 \times \frac{8}{16}\frac{4}{8}\frac{2}{4} + 4 \times (1 - \frac{8}{16} - \frac{8}{16}\frac{4}{16} - \frac{8}{16}\frac{4}{8}\frac{2}{4}) = 1 \times 0.5 + 2 \times 0.25 + 3 \times 0.125 + 4 \times 0.125 = 15/8 = 1.875$.

2. Generalize the above idea to infinitely many tosses. For expected number of tosses, you will get an arithmetic-geometric progression that will sum to 2 tosses on average.