Problems for Discussion 6

November 6, 2019

Exercise 1.
We say a matrix $M_{n\times n}$ is diagonally dominant if for all $1 \leq i \leq n$, $M_{ii} \geq \sum_{j\neq i} |M_{ij}|$, and it is strictly diagonally dominant if the inequality is strict.

Show that if a matrix M is symmetric and strictly diagonally dominant, then it is non-singular.

(See solution below for additional remarks.)

Exercise 2.
In the previous problem, for a self-adjoint matrix, we concluded that if the Gershgorin disks are constrained to positive part of real line, then the matrix is positive definite. Is the converse true?

Exercise 3.
(Puzzle problem from last year) Do there exist polynomials $p(x), q(x), r(y), s(y)$ (of any degree) such that $1 + xy + x^2y^2 = p(x)r(y) + q(x)s(y)$?

Hints/Solutions

Solution 1.

In fact, any such matrix is positive definite, and so non-singular (invertible/full rank) as well. Apply the Gershgorin disk theorem to see all the disks have real part positive. Moreover, because M is symmetric, the eigenvalues must be real, and so the disks are in fact intervals, all contained in $\mathbb{R}_{>0}$. So, all the eigenvalues are positive, and the matrix is positive definite.

Remark 1. The same proof shows that if a matrix M is symmetric and diagonally dominant, then it is positive semidefinite. This means that for any vector $x \in \mathbb{R}^n, x^T M x \geq 0$.

Remark 2. While it doesn't make sense to talk of positive semidefiniteness for non-symmetric matrices, we can still ask if $x^T M x \geq 0$ condition is true if we just assume M to be diagonally dominant and not necessarily symmetric.

The answer is no. We can see that the symmetry (self-adjointness) is really needed by giving an M, x pair such that M is diagonally dominant but $x^T M x < 0$. One example is $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$.

Solution 2.

No, to construct a counterexample of dimensions 2×2, choose the first row so that the disk has negative values too. Let this be $[1 \;\; 2]$. For the matrix to be symmetric, we know that first entry of second row is 2. Choose the second entry of matrix to be any number larger than 4 to ensure both eigenvalues are positive (prove this). So, an example would be $\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$

Hint 3.

For any value of y, $p(x)r(y) + q(x)s(y)$ lies in the span of $p(x)$ and $q(x)$, and so we cannot generate more than 2 linearly independent polynomials in x by different assignments to y.