Problems for Discussion 5

October 30, 2019

Exercise 1.
Recall that every inner product induces a natural norm on the vector space defined as $||x|| = \sqrt{\langle x, x\rangle}$. Show that ℓ_p norm defined in class cannot be induced by an inner produce for $1 \leq p < 2$ or $p > 2$. (See footnote on page 2 of Lecture 9.)

Exercise 2.
For any linear operator φ on n-dimensional vector space V, the following are equivalent:

(a) φ is an isometry.

(b) φ is unitary, that is $\varphi^*\varphi = \mathrm{id}$.

(c) φ maps orthonormal bases to orthonormal bases.

Exercise 3.
(Puzzle problem 2 from Lecture 1) Show that to cover the non-zero vertices of the n-dimensional hypercube $\{0,1\}^n$ using hyperplanes not passing through 0^n, we will need at least n hyperplanes.

Hints

Hint 1.
Any norm $||.||$ induced by an inner product must satisfy the parallelogram law:

$$||a-b|| + ||a+b|| = 2||a|| + 2||b||$$

Hint 2.
(a) $\implies$ (b) : $||v|| = ||\varphi(v)||$ means $\langle v, v\rangle = \langle \varphi v, \varphi v\rangle = \langle \varphi^* \varphi(v), v\rangle$, or $\langle (\varphi^*\varphi - \mathrm{id})(v), v\rangle = 0$ for every $v \in V$.

From here, conclude that $\varphi^*\varphi - \mathrm{id} = 0$. Here, you will crucially use the self-adjointness of $\varphi^*\varphi - \mathrm{id}$, as in general this statement is not true. Consider the rotation map ψ in $\mathbb{R}^2$ that rotates every vector by 90 degrees. Clearly, $\langle \psi(v), v\rangle = 0$ for every v but $\psi \neq 0$.

(b) $\implies$ (c): φ being unitary implies $\langle \varphi(u), \varphi(v)\rangle = \langle u, v\rangle$.

(c) $\implies$ (a): Easy.

Hint 3.
Consider the vector space V of all functions mapping $\{0,1\}^n$ to $\mathbb{R}$. The dimension of this space is 2^n.

Show that the functions $f_I(x_1, x_2, \cdots, x_n) = \prod_{i\in I} x_i$ are linearly independent in V. As these are 2^n in number, this must be a basis.

The function that is 1 on 0^n and 0 everywhere else has the unique representation $(1-x_1)(1-x_2)\cdots(1-x_n)$ with degree n. Use the hyperplanes to get another representation for the same function as a multilinear polynomial, with degree equal to number of hyperplanes. Because the representation must be unique, number of hyperplanes must be at least n.