Problems for Discussion 4

October 23, 2019

Exercise 1.
Let V be a finite dimensional inner product space with $\dim(V) = n$. Let $\alpha, \beta : V \to V$ be two self-adjoint linear operators. Furthermore, suppose that β is positive semidefinite. Show that for $k \in [n]$, $\lambda_k(\alpha + \beta) \geq \lambda_k(\alpha)$.

Exercise 2.
(1097 Golan's Book) Let V be an inner product space finitely generated over $\mathbb{C}$ and let $\alpha : V \to V$ be a positive definite self-adjoint full rank linear operator. Show that $\langle(\alpha + \alpha^{-1})v, v\rangle \geq 2\langle v, v\rangle$ for all $v \in V$. Use this to show that the minimum eigenvalue of $\alpha + \alpha^{-1}$ is at least 2.

Exercise 3.
Do the Cholesky Decomposition on $\begin{bmatrix} 4 & -6 & -2 \\ -6 & 13 & 1 \\ -2 & 1 & 11 \end{bmatrix}$.

Solutions

Solution 1.

The key observation is that Rayleigh quotient is linear, so that $\mathcal{R}_{\alpha+\beta}(v) = \mathcal{R}_\alpha(v) + \mathcal{R}_\beta(v)$ for all $v \in V \backslash \{0_V\}$. Then β being positive semidefinite means that $\mathcal{R}_{\alpha+\beta}(v) \geq \mathcal{R}_\alpha(v)$ for all $v \in V \backslash \{0_V\}$.

Next, we need to ensure that $\lambda_k(\alpha+\beta) = \max\limits_{\substack{S \subseteq V \\ \dim S = k}} \min\limits_{v \in S \backslash \{0\}} \mathcal{R}_{\alpha+\beta}(v) \geq \max\limits_{\substack{S \subseteq V \\ \dim S = k}} \min\limits_{v \in S \backslash \{0\}} \mathcal{R}_\alpha(v) = \lambda_k(\alpha)$.

To do this, let S be arbitrary, and choose $v^* \in S$ that minimizes $\mathcal{R}_{\alpha+\beta}(v)$. Then

$$\min_{v \in S \backslash \{0\}} \mathcal{R}_{\alpha+\beta}(v) = \mathcal{R}_{\alpha+\beta}(v^*) \geq \mathcal{R}_\alpha(v^*) \geq \min_{v \in S \backslash \{0\}} \mathcal{R}_\alpha(v)$$

Note that the above holds for all subspaces S. Now choose S^* that maximizes $\min\limits_{v \in S \backslash \{0\}} \mathcal{R}_\alpha(v)$, and repeat the above argument to get

$$\max_{\substack{S \subseteq V \\ \dim S = k}} \min_{v \in S \backslash \{0\}} \mathcal{R}_{\alpha+\beta}(v) \geq \max_{\substack{S \subseteq V \\ \dim S = k}} \min_{v \in S \backslash \{0\}} \mathcal{R}_\alpha(v)$$

Solution 2.

If the eigenvalues of α are $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the eigenvalues of α^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$, and the corresponding eigenvectors are same.

Because α is self-adjoint, it is orthogonally diagonalizable, and let $v_1, v_2, \cdots, v_n$ be orthonormal set of eigenvectors that forms a basis, corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then every $v = \sum_{i=1}^n c_i v_i$, and $\alpha(v) = \sum_{i=1}^n c_i \lambda_i v_i$, and $\alpha^{-1}(v) = \sum_{i=1}^n \frac{c_i}{\lambda_i} v_i$. Together,

$$\begin{aligned}
\langle (\alpha + \alpha^{-1})(v), v \rangle &= \langle \sum_{i=1}^n (\lambda_i + \frac{1}{\lambda_i}) c_i v_i, \sum_{i=1}^n c_i v_i \rangle \\
&= \sum_{i=1}^n (\lambda_i + \frac{1}{\lambda_i}) c_i^2 \\
&\geq \sum_{i=1}^n 2c_i^2 = 2\langle v, v \rangle
\end{aligned}$$

To see why minimum eigenvalue of $\alpha + \alpha^{-1}$ is at least 2, note that the Rayleigh quotients are lower bounded by 2 for every vector.

Solution 3.

$$\begin{bmatrix} 4 & -6 & -2 \\ -6 & 13 & 1 \\ -2 & 1 & 11 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 0 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -3 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$