Problems for Discussion 3

October 16, 2019

Exercise 1.
Consider the vector space V with a basis $\{v_1, v_2, \cdots, v_n\}$. Show that if two vectors $x, y \in V$ are such that $\langle x, v_i \rangle = \langle y, v_i \rangle$ for all $i \in [n]$, then $x = y$.

Exercise 2.
Let U, V, W be finite dimensional vector spaces, and let $\alpha : U \to V$ and $\beta : V \to W$ be linear transformations. Let $\beta \circ \alpha : U \to W$ be the composite operator defined as $\beta \circ \alpha(u) = \beta(\alpha(u))$. Show that

$$\dim \ker \beta \circ \alpha \leq \dim \ker \alpha + \dim \ker \beta$$

Exercise 3.
Let V be a vector space of dimension n over the real field. Show that any linear operator $\varphi : V \to V$ can be written as the sum of two full-rank (or trivial kernel) operators.
In particular, this means that any $n \times n$ matrix can be written as the sum of two non-singular matrices.

Solutions

Solution 1.
Let $z = x - y$, so that for every $i \in [n]$,

$$\begin{aligned}
\langle z, v_i \rangle = \langle x - y, v_i \rangle &= \overline{\langle v_i, x - y \rangle} \\
&= \overline{\langle v_i, x \rangle - \langle v_i, y \rangle} \\
&= \overline{\langle v_i, x \rangle} + \overline{(-1)\langle v_i, y \rangle} \\
&= \langle x, v_i \rangle + (-1)\langle y, v_i \rangle \\
&= \langle x, v_i \rangle - \langle y, v_i \rangle = 0
\end{aligned}$$

Now, because $\{v_1, v_2, \cdots, v_n\}$ is a basis, there exist c_i such that $z = \sum\limits_{i=1}^{n} c_i v_i$.

This means $\langle z, z \rangle = \langle z, \sum_{i=1}^{n} c_i v_i \rangle = \sum_{i=1}^{n} c_i \langle z, v_i \rangle = 0$. From the properies of inner products, it follows that $z = 0$, or $x = y$.

Solution 2.
Because any $u \in \ker \alpha$ has $\alpha(u) = 0_V$, it also has $\beta \circ \alpha(u) = 0_W$, and so $u \in \ker \beta \circ \alpha$. This means $\ker \alpha \subseteq \ker \beta \circ \alpha$.

Let $\gamma : \ker \beta \circ \alpha \to V$ be a linear operator defined as $\gamma(u) = \alpha(u)$. From the previous observation, it follows that $\ker \gamma = \ker \alpha$. Applying the rank-nullity theorem on γ,

$$\begin{aligned}
\dim \ker \beta \circ \alpha &= \dim \ker \gamma + \dim \operatorname{im} \gamma \\
&= \dim \ker \alpha + \dim \operatorname{im} \gamma \\
\dim \ker \beta \circ \alpha - \dim \ker \alpha &= \dim \operatorname{im} \gamma
\end{aligned}$$

Thus to show that $\dim \operatorname{im} \gamma = \dim \ker \beta \circ \alpha - \dim \ker \alpha \leq \dim \ker \beta$, it suffices to show that $\operatorname{im} \gamma \subseteq \ker \beta$.

Let $v \in \operatorname{im} \gamma$. By definition, there exists some $u \in \ker \beta \circ \alpha$ such that $\gamma(u) = \alpha(u) = v$. But then, $\beta(v) = \beta \circ \alpha(u) = 0_W$ because $u \in \ker \beta \circ \alpha$. This means $v \in \ker \beta$, and so it follows that $\operatorname{im} \gamma \subseteq \ker \beta$.

Solution 3.
Note that for any real α, $\varphi = \varphi_1 + \varphi_2$ is a valid decomposition where $\varphi_1 = \varphi - \alpha\,\mathrm{id}$ and $\varphi_1 = \alpha\,\mathrm{id}$, where id is the identity operator mapping every vector in V to itself.

Clearly, φ_2 has a trivial kernel for any non-zero α. For φ_1 to be full rank (equivalently, have trivial kernel), we must have that $\varphi_1(v) \neq 0_V$, or $\varphi(v) \neq \alpha v$, for any $v \neq 0_V$.

Consider any $\alpha \notin \mathit{spec}(\varphi)$. Then for any non-zero vector v in V, $\varphi(v) \neq \alpha v$, and so this choice of α indeed makes φ_1 full rank.

Finally, we choose α to be a real number that is neither 0 nor an eigenvalue. Because there are at most n distinct eigenvalues (because distinct eigenvalues have linearly independent eigenvectors and we are working in an n-dimensional space), we rule out at most $n + 1$ real numbers, and any real other than these will give us the required decomposition into full rank operators.

The matrix result follows by looking at matrices as operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ and then the id operator corresponds to the identity matrix.