Problems for Discussion 1

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Exercise 1.
Let p be a prime. Show that the set $\{0, 1, \cdots, p-1\}$ associated with operations $+_{\mathbb{F}}$ and $._{\mathbb{F}}$ defined below is a field.

(i) $a +_{\mathbb{F}} b = (a + b) \mod p$
(ii) $a ._{\mathbb{F}} b = (a.b) \mod p$
This field denoted as $\mathbb{F}_p$. Show that this is not a field if p is composite.

Exercise 2.
Consider the Fibonacci vector space defined in class over the real field. The set of vectors is given by

$$\mathsf{Fib} = \{f \in \mathbb{R}^{\mathbb{N}} : f(n) = f(n-1) + f(n-2),\ \forall n \geq 2\}$$

Each element of Fib can be viewed either as a function from natural numbers to reals, or as an infinite sequence of reals.

(i) What is the additive identity in this vector space?
(ii) What is the additive inverse for an element of Fib?
(iii) Show that the functions/sequences $(1, 1, 2, 3, 5, 8, \cdots)$ and $(1, 2, 3, 5, 8, \cdots)$ are linearly independent in Fib.

Exercise 3.
Let $n \geq 2$, and let V be a vector space over $\mathbb{R}$. Let $v_1, v_2, \cdots, v_n$ be linearly independent vectors in V.

(i) Let

$$\begin{aligned} u_1 &= v_1 + v_2 + \cdots + v_{n-2} + v_{n-1} \\ u_2 &= v_2 + v_3 + \cdots + v_{n-1} + v_n \\ &\vdots \\ u_n &= v_n + v_1 + \cdots + v_{n-3} + v_{n-2} \end{aligned}$$

Show that $u_1, u_2, \cdots, u_n$ are linearly independent.
(ii) Let

$$\begin{aligned} u_1 &= v_1 + v_2 \\ u_2 &= v_2 + v_3 \\ &\vdots \\ u_n &= v_n + v_1 \end{aligned}$$

Show that $u_1, u_2, \cdots, u_n$ can be linearly dependent.
(iii) Let $u_1, u_2, \cdots, u_n$ be as in (ii). Show that $u_1, u_2, \cdots, u_n$ are linearly independent if n is odd.
(iv) Let $1 \leq k \leq n$. Let

$$\begin{aligned}
u_1 &= v_1 + v_2 + \cdots + v_{k-1} + v_k \\
u_2 &= v_2 + v_3 + \cdots + v_k + v_{k+1} \\
&\vdots \\
u_n &= v_n + v_1 + \cdots + v_{k-2} + v_{k-1}
\end{aligned}$$

Show that $u_1, u_2, \cdots, u_n$ are linearly independent iff $gcd(k, n) = 1$.