Mathematical Toolkit	**Autumn 2019**

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1 Eigenvalues and eigenvectors

Definition 1.1 *Let V be a vector space over the field $\mathbb{F}$ and let $\varphi : V \to V$ be a linear transformation. $\lambda \in \mathbb{F}$ is said to be an* eigenvalue *of φ if there exists $v \in V \setminus \{0_V\}$ such that $\varphi(v) = \lambda \cdot v$. Such a vector v is called an* eigenvector *corresponding to the eigenvalue λ. The set of eigenvalues of φ is called its* spectrum*:*

$$\operatorname{spec}(\varphi) = \{\lambda \mid \lambda \text{ is an eigenvalue of } \varphi\} \ .$$

Example 1.2 *Consider the following transformations:*

- *Differentiation is a linear transformation on the class of (say) infinitely differentiable real-valued functions over $[0,1]$ (denoted by $C^\infty([0,1],\mathbb{R})$). Each function of the form $c \cdot \exp(\lambda x)$ is an eigenvector with eigenvalue λ. If we denote the transformation by φ_0, then $\operatorname{spec}(\varphi_0) = \mathbb{R}$.*
- *We can also consider the transformation $\varphi_1 : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by differentiation i.e., for any polynomial $P \in \mathbb{R}[x]$, $\varphi_1(P) = dP/dx$. Note that now the only eigenvalue is 0, and thus $\operatorname{spec}(\varphi) = \{0\}$.*
- *Consider the transformation $\varphi_{\text{left}} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$. Any geometric progression with common ratio r is an eigenvector of φ_{left} with eigenvalue r (and these are the only eigenvectors for this transformation).*

Example 1.3 *It can also be the case that $\operatorname{spec}(\varphi) = \emptyset$, as witnessed by the rotation matrix*

$$M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

when viewed as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Proposition 1.4 *Let $U_\lambda = \{v \in V \mid \varphi(v) = \lambda \cdot v\}$. Then for each $\lambda \in \mathbb{F}$, U_λ is a subspace of V.*

Note that $U_\lambda = \{0_V\}$ if λ is not an eigenvalue. The dimension of this subspace is called the geometric multiplicity of the eigenvalue λ.

Proposition 1.5 *Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of φ with associated eigenvectors $v_1, \ldots, v_k$. Then the set $\{v_1, \ldots, v_k\}$ is linearly independent.*

Definition 1.6 *A transformation $\varphi : V \to V$ is said to be* diagonalizable *if there exists a basis of V comprising of eigenvectors of φ.*

Exercise 1.7 *Recall that* $\mathrm{Fib} = \left\{f \in \mathbb{R}^{\mathbb{N}} \mid f(n) = f(n-1) + f(n-2)\ \forall n \geq 2\right\}$. *Show that $\varphi_{\mathrm{left}} : \mathrm{Fib} \to \mathrm{Fib}$ is diagonalizable. Express the sequence by $f(0) = 1, f(1) = 1$ and $f(n) = f(n-1) + f(n-2)\ \forall n \geq 2$ (known as Fibonacci numbers) as a linear combination of eigenvectors of φ_{left}.*

2 Inner Products

For the discussion below, we will take the field $\mathbb{F}$ to be $\mathbb{R}$ or $\mathbb{C}$ since the definition of inner products needs the notion of a "magnitude" for a field element (these can be defined more generally for subfields of $\mathbb{R}$ and $\mathbb{C}$ known as Euclidean subfields, but we shall not do so here).

Definition 2.1 *Let V be a vector space over a field $\mathbb{F}$ (which is taken to be $\mathbb{R}$ or $\mathbb{C}$). A function $\mu : V \times V \to \mathbb{F}$ is an inner product if*

- *The function $\mu(u, \cdot) : V \to \mathbb{F}$ is a linear transformation for every $u \in V$.*
- *The function satisfies $\mu(u, v) = \overline{\mu(v, u)}$.*
- *$\mu(v, v) \in \mathbb{R}_{\geq 0}$ for all $v \in V$ and is 0 only for $v = 0_V$.*

We write the inner product corresponding to μ as $\langle u, v \rangle$.

Strictly speaking, the inner product should be written as $\langle u, v \rangle_\mu$ but we usually omit the μ when the function is clear from context (or we are referring to an arbitrary inner product).

Example 2.2 *The following are all examples of inner products:*

- *The function $\int_{-1}^{1} f(x)g(x)dx$ for $f, g \in C([-1, 1], \mathbb{R})$ (space of continuous functions from $[-1, 1]$ to $\mathbb{R}$).*
- *The function $\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}dx$ for $f, g \in C([-1, 1], \mathbb{R})$.*

- *For* $x, y \in \mathbb{R}^2$, $\langle x, y \rangle = x_1 y_1 + x_2 y_2$ *is the usual inner product. Check that* $\langle x, y \rangle = 2x_1 y_1 + x_2 y_2 + x_1 y_2 / 2 + x_2 y_1 / 2$ *also defines an inner product.*

Exercise 2.3 *Let* $C > 4$*. Check that*

$$\mu(f, g) \; = \; \sum_{n=0}^{\infty} \frac{f(n) \cdot g(n)}{C^n}$$

defines an inner product on the space Fib.