

Lecture 9: October 30, 2018

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1 Applications of SVD: least squares approximation

We discuss another application of singular value decomposition (SVD) of matrices. Let $a_1, \dots, a_n \in \mathbb{R}^d$ be points which we want to fit to a low-dimensional subspace. The goal is to find a subspace S of \mathbb{R}^d of dimension at most k to minimize $\sum_{i=1}^n (\text{dist}(a_i, S))^2$, where $\text{dist}(a_i, S)$ denotes the distance of a_i from the closest point in S . We first prove the following.

Claim 1.1 *Let u_1, \dots, u_k be an orthonormal basis for S . Then*

$$(\text{dist}(a_i, S))^2 = \|a_i\|_2^2 - \sum_{j=1}^k \langle a_i, u_j \rangle^2.$$

Thus, the goal is to find a set of k orthonormal vectors u_1, \dots, u_k to maximize the quantity $\sum_{i=1}^n \sum_{j=1}^k \langle a_i, u_j \rangle^2$. Let $A \in \mathbb{R}^{n \times d}$ be a matrix with the i^{th} row equal to a_i^T . Then, we need to find orthonormal vectors u_1, \dots, u_k to maximize $\|Au_1\|_2^2 + \dots + \|Au_k\|_2^2$. We will prove the following.

Proposition 1.2 *Let v_1, \dots, v_r be the right singular vectors of A corresponding to singular values $\sigma_1 \geq \dots \geq \sigma_r > 0$. Then, for all $k \leq r$ and all orthonormal sets of vectors u_1, \dots, u_k*

$$\|Au_1\|_2^2 + \dots + \|Au_k\|_2^2 \leq \|Av_1\|_2^2 + \dots + \|Av_k\|_2^2$$

Thus, the optimal solution is to take $S = \text{Span}(v_1, \dots, v_k)$. We prove the above by induction on k . For $k = 1$, we note that

$$\|Au_1\|_2^2 = \langle A^T Au_1, u_1 \rangle \leq \max_{v \in \mathbb{R}^d \setminus \{0\}} \mathcal{R}_{A^T A}(v) = \sigma_1^2 = \|Av_1\|_2^2.$$

To prove the induction step for a given $k \leq r$, define

$$V_{k-1}^\perp = \left\{ v \in \mathbb{R}^d \mid \langle v, v_i \rangle = 0 \ \forall i \in [k-1] \right\}.$$

First prove the following claim.

Claim 1.3 Given an orthonormal set u_1, \dots, u_k , there exist orthonormal vectors u'_1, \dots, u'_k such that

- $u'_k \in V_{k-1}^\perp$.
- $\text{Span}(u_1, \dots, u_k) = \text{Span}(u'_1, \dots, u'_k)$.
- $\|Au_1\|_2^2 + \dots + \|Au_k\|_2^2 = \|Au'_1\|_2^2 + \dots + \|Au'_k\|_2^2$.

Proof: We only provide a sketch of the proof here. Let $S = \text{Span}(\{u_1, \dots, u_k\})$. Note that $\dim(V_{k-1}^\perp) = d - k + 1$ (why?) and $\dim(S) = k$. Thus,

$$\dim(V_{k-1}^\perp \cap S) \geq k + (d - k + 1) - d = 1.$$

Hence, there exists $u'_k \in V_{k-1}^\perp \cap S$ with $\|u'_k\| = 1$. Completing this to an orthonormal basis of S gives orthonormal u'_1, \dots, u'_k with the first and second properties. We claim that this already implies the third property (why?). ■

Thus, we can assume without loss of generality that the given vectors u_1, \dots, u_k are such that $u_k \in V_{k-1}^\perp$. Hence,

$$\|Au_k\|_2^2 \leq \max_{\substack{v \in V_{k-1}^\perp \\ \|v\|=1}} \|Av\|_2^2 = \sigma_k^2 = \|Av_k\|_2^2.$$

Also, by the inductive hypothesis, we have that

$$\|Au_1\|_2^2 + \dots + \|Au_{k-1}\|_2^2 \leq \|Av_1\|_2^2 + \dots + \|Av_{k-1}\|_2^2,$$

which completes the proof. The above proof can also be used to prove that SVD gives the best rank k approximation to the matrix A in Frobenius norm. We will see this in the next homework.

2 Bounding the eigenvalues: Gershgorin Disc Theorem

We will now see a simple but extremely useful bound on the eigenvalues of a matrix, given by the Gershgorin disc theorem. Many useful variants of this bound can also be derived from the observation that for any invertible matrix S , the matrices $S^{-1}MS$ and M have the same eigenvalues (prove it!).

Theorem 2.1 (Gershgorin Disc Theorem) Let $M \in \mathbb{C}^{n \times n}$. Let $R_i = \sum_{j \neq i} |M_{ij}|$. Define the set

$$\text{Disc}(M_{ii}, R_i) := \{x \mid x \in \mathbb{C}, |x - M_{ii}| \leq R_i\}.$$

If λ is an eigenvalue of M , then

$$\lambda \in \bigcup_{i=1}^n \text{Disc}(M_{ii}, R_i).$$

Proof: Let $z \in \mathbb{C}^n$ be an eigenvector corresponding to the eigenvalue λ . Let $i_0 = \text{argmax}_{i \in [n]} \{|z_i|\}$. Since z is an eigenvector, we have

$$Mz = \lambda z \quad \Rightarrow \quad \forall i \in [n] \quad \sum_{j=1}^n M_{ij} z_j = \lambda z_i.$$

In particular, we have that for $i = i_0$,

$$\sum_{j=1}^n M_{i_0 j} z_j = \lambda z_{i_0} \quad \Rightarrow \quad \sum_{j=1}^n M_{i_0 j} \frac{z_j}{z_{i_0}} = \lambda \quad \Rightarrow \quad \sum_{j \neq i_0} M_{i_0 j} \frac{z_j}{z_{i_0}} = \lambda - M_{i_0 i_0}.$$

Thus, we have

$$|\lambda - M_{i_0 i_0}| \leq \sum_{j \neq i_0} |M_{i_0 j}| \cdot \left| \frac{z_j}{z_{i_0}} \right| \leq \sum_{j \neq i_0} |M_{i_0 j}| = R_{i_0}.$$

■

2.1 An application to compressed sensing

The Gershgorin disc theorem is quite useful in compressed sensing, to ensure what is known as the “Restricted Isometry Property” for the measurement matrices.

Definition 2.2 A matrix $A \in \mathbb{R}^{k \times n}$ is said to have the restricted isometry property with parameters (s, δ_s) if

$$(1 - \delta_s) \cdot \|x\|^2 \leq \|Ax\|^2 \leq (1 + \delta_s) \cdot \|x\|^2$$

for all $x \in \mathbb{R}^n$ which satisfy $|\{i \mid x_i \neq 0\}| \leq s$.

Thus, we want the transformation A to be approximately norm preserving for all *sparse* vectors x . This can of course be ensured for all x by taking $A = \text{id}$, but we require $k \ll n$ for the applications in compressed sensing. In general, the restricted isometry property is NP-hard to verify and can thus also be hard to reason about for a given matrix. The Gershgorin Disc Theorem lets us derive a much easier condition which is sufficient to ensure the restricted isometry property.

Definition 2.3 Let $A \in \mathbb{R}^{k \times n}$ be such that $\|A^{(i)}\| = 1$ for each column $A^{(i)}$ of A . Define the coherence of A as

$$\mu(A) = \max_{i \neq j} \left| \langle A^{(i)}, A^{(j)} \rangle \right|.$$

We will prove the following

Proposition 2.4 Let $A \in \mathbb{R}^{k \times n}$ be such that $\|A^{(i)}\| = 1$ for each column $A^{(i)}$ of A . Then, for any s , the matrix A has the restricted isometry property with parameters $(s, (s-1)\mu(A))$.

Note that the bound becomes meaningless if $s \geq 1 + \frac{1}{\mu(A)}$. However, the above proposition shows that it may be sufficient to bound $\mu(A)$ (which is also easier to check in practice).

Proof: Consider any x such that $|\{i \mid x_i \neq 0\}| \leq s$. Let S denote the support of x i.e., $S = \{i \mid x_i \neq 0\}$. Let A_S denote the $k \times |S|$ submatrix where we only keep the columns corresponding to indices in S . Let x_S denote x restricted to the non-zero entries. Then

$$\|Ax\|^2 = \|A_S x_S\|^2 = \langle A_S^T A_S x_S, x_S \rangle.$$

Thus, it suffices to bound the eigenvalues of the matrix $A_S^T A_S$. Note that $(A_S)_{ij} = \langle A^{(i)}, A^{(j)} \rangle$. Thus the diagonal entries are 1 and the off-diagonal entries are bounded by $\mu(A)$ in absolute value. By the Gershgorin Disc Theorem, for any eigenvalue λ of A , we have

$$|\lambda - 1| \leq (s-1) \cdot \mu(A).$$

Thus, we have

$$(1 - (s-1) \cdot \mu(A)) \cdot \|x\|^2 \leq \|Ax\|^2 \leq (1 + (s-1) \cdot \mu(A)) \cdot \|x\|^2,$$

as desired. ■

The theorem is also very useful for bounding how much the eigenvalues of matrix change due to a perturbation. We will see an example of this in the homework.