**Mathematical Toolkit** **Autumn 2018**

# Lecture 6: October 18, 2018

Lecturer: Madhur Tulsiani

## 1 Self-adjoint transformations

**Definition 1.1** *A linear transformation $\varphi : V \to V$ is called* self-adjoint *if $\varphi = \varphi^*$. Linear transformations from a vector space to itself are called* linear operators.

**Example 1.2** *The transformation represented by matrix $A \in \mathbb{C}^{n \times n}$ is self-adjoint if $A = \overline{A^T}$. Such matrices are called Hermitian matrices.*

**Proposition 1.3** *Let $V$ be an inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then*

- *All eigenvalues of $\varphi$ are real.*
- *If $\{w_1, \ldots, w_n\}$ are eigenvectors corresposnding to distinct eigenvalues then they are mutually orthogonal.*

We will use the following statement (which we'll prove later) to prove the spectral theorem.

**Proposition 1.4** *Let $V$ be a finite-dimensional inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then $\varphi$ has at least one eigenvalue.*

Using the above proposition, we will prove the spectral theorem below for finite dimensional vector spaces. The proof below can also be made to work for Hilbert spaces (using the axiom of choice). The above proposition, which gives the existence of an eigenvalue is often proved differently for finite and infinite-dimensional spaces, and the proof for infinite-dimensional Hilbert spaces requires additional conditions on the operator $\varphi$. We first prove the spectral theorem assuming the above proposition.

**Proposition 1.5 (Real spectral theorem)** *Let $V$ be a finite-dimensional inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then $\varphi$ is orthogonally diagonalizable.*

**Proof:** By induction on the dimension of $V$. Let $\dim(V) = 1$. Then by the previous proposition $\varphi$ has at least one eigenvalue, and hence at least one eigenvector, say $w$. Then $w/\|w\|$ is a unit vector which forms a basis for $V$.

Let $\dim(V) = k+1$. Again, by the previous proposition $\varphi$ has at least one eigenvector, say $w$. Let $W = \text{Span}(\{w\})$ and let $W^{\perp} = \{v \in V \mid \langle v, w\rangle = 0\}$. Check the following:

- $W^{\perp}$ is a subspace of $V$.
- $\dim(W^{\perp}) = k$.
- $W^{\perp}$ is invariant under $\varphi$ i.e., $\forall v \in W^{\perp}$, $\varphi(v) \in W^{\perp}$.

Thus, we can consider the operator $\varphi' : W^{\perp} \to W^{\perp}$ defined as

$$\varphi'(v) \;:=\; \varphi(v) \qquad \forall v \in W^{\perp}.$$

Then, $\varphi'$ is a self-adjoint (check!) operator defined on the $k$-dimensional space $W^{\perp}$. By the induction hypothesis, there exists an orthonormal basis $\{w_1, \ldots, w_k\}$ for $W^{\perp}$ such that each $w_i$ is an eigenvector of $\varphi$. Thus $\left\{w_1, \ldots, w_k, \frac{w}{\|w\|}\right\}$ is an orthonormal basis for $V$, comprising of eigenvectors of $\varphi$. ■

## 2 Existence of eigenvalues

We now prove Proposition 1.4, which shows that a self-adjoint operator must have at least one eigenvalue. Let us assume for now that $V$ is an inner product space over $\mathbb{C}$. As was observed in class, in this case we don't need self-adjointness to guarantee an eigenvalue. We thus prove the following more general result

**Proposition 2.1** *Let $V$ be a finite dimensional inner product space over $\mathbb{C}$ and let $\varphi : V \to V$ be a linear operator. Then $\varphi$ has at least one eigenvalue.*

**Proof:** Let $\dim(V) = n$. Let $v \in V \setminus 0_V$ be any non-zero vector. Consider the set of $n+1$ vectors $\{v, \varphi(v), \ldots, \varphi^n(v)\}$. Since the dimension of $V$ is $n$, there must exists $c_0, \ldots, c_n \in \mathbb{C}$ such that

$$c_0 \cdot v + c_1 \cdot \varphi(v) + \cdots + c_n \varphi^n(v) \;=\; 0_V.$$

We assume above that $c_n \neq 0$, otherwise we can only consider the sum to the largest $i$ such that $c_i \neq 0$. Let $P(x)$ denote the polynomial $c_0 + c_1 x + \cdots + c_n x^n$. Then the above can be written as $(P(\varphi))(v) = 0$, where $P(\varphi) : V \to V$ is a linear operator defined as

$$P(\varphi) \;:=\; c_0 \cdot \mathsf{id} + c_1 \cdot \varphi + \cdots + c_n \varphi^n,$$

with $\mathsf{id}$ used to denote the identity operator. Since $P$ is a degree-$n$ polynomial over $\mathbb{C}$, it can be factored into $n$ linear factors, and we can write $P(x) = c_n \prod_{i=1}^{n} (x - \lambda_i)$ for $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. This means that we can write

$$P(\varphi) \;=\; c_n(\varphi - \lambda_n \cdot \mathsf{id}) \cdots (\varphi - \lambda_1 \cdot \mathsf{id})\,.$$

Let $w_0 = v$ and define $w_i = \varphi(w_{i-1}) - \lambda_i \cdot w_{i-1}$ for $i \in [n]$. Note that $w_0 = v \neq 0_V$ and $w_n = P(\varphi)(v) = 0_V$. Let $i^*$ denote the largest index $i$ such that $w_i \neq 0_V$. Then, we have

$$0_V \;=\; w_{i^*+1} \;=\; \varphi(w_{i^*}) - \lambda_{i^*+1} \cdot w_{i^*}\,.$$

This implies that $w_{i^*}$ is an eigenvector with eigenvalue $\lambda_{i^*+1}$. ■

To prove Proposition 1.4 using this, we note that $\varphi = \varphi^*$ implies the eigenvalue found by the above proposition must are real.

**Exercise 2.2** *Use the fact that the eigenvalues of a self-adjoint operator are real to prove Proposition 1.4 even when $V$ is an inner product space over $\mathbb{R}$.*