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1 Threshold Phenomena in Random Graphs

We consider a model of Random Graphs by Erdős and Rényi [ER60]. To generate a random graph with n vertices, for every pair of vertices $\{i, j\}$, we put an edge independently with probability p . This model is denoted by $\mathcal{G}_{n,p}$.

Let G be a random $\mathcal{G}_{n,p}$ graph and let H be any fixed graph (on some constant number of vertices independent of n). We will be interested in understanding the probability that G contains a copy of H . We start by computing this when H is K_4 , the clique on 4 vertices.

Definition 1.1 We define k -clique to be a fully connected graph with k vertices.

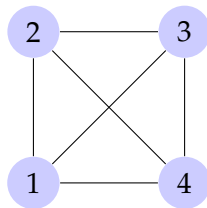


Figure 1: 4-Clique

As a convention, we will count a permutation of a copy of K_4 as the *same* copy. We define the random variable

$$Z = \text{number of copies of } K_4 \text{ in } G = \sum_C X_C,$$

where C ranges over all subsets of V of size 4 and the random variable X_C is defined as

$$X_C = \begin{cases} 1 & \text{if all pair of vertices in the set } C \text{ have an edge in between them} \\ 0 & \text{otherwise} \end{cases}.$$

We have $\mathbb{E}[X_C] = p^6$, since the probability of connecting all 4 vertices (using 6 edges) in the 4-tuple is p^6 . So we have the expectation of Z :

$$\mathbb{E}[Z] = \sum_C \mathbb{E}[X_C] = \binom{n}{4} \cdot p^6$$

We observe that

$$\mathbb{E}[Z] \rightarrow 0 \text{ when } p \ll n^{-2/3} \quad \text{and} \quad \mathbb{E}[Z] \rightarrow \infty \text{ when } p \gg n^{-2/3}.$$

Here, by $p \ll n^{-2/3}$, we mean that $\lim_{n \rightarrow \infty} (p/n^{-2/3}) = 0$ and $p \gg n^{-2/3}$ is defined similarly. We will prove that there is in fact a threshold phenomenon in the probability that G contains a copy of K_4 . When $p \ll n^{-2/3}$, the probability that a random graph G generated according to model $\mathcal{G}_{n,p}$ contains a copy of K_4 , goes to 0 as $n \rightarrow \infty$. On the other hand, when $p \gg n^{-2/3}$, this probability tends to 1.

Theorem 1.2 *Let G be generated randomly according to the model $\mathcal{G}_{n,p}$ graph. We have that:*

- If $p \ll n^{-2/3}$, then $\mathbb{P}[G \text{ contains a copy of } K_4] \rightarrow 0$ as $n \rightarrow \infty$.
- If $p \gg n^{-2/3}$, then $\mathbb{P}[G \text{ contains a copy of } K_4] \rightarrow 1$ as $n \rightarrow \infty$.

Proof: As above, we define the random variable Z ,

$$Z = \text{number of copies of } K_4 \text{ in } G = \sum_C X_C.$$

The case when $p \ll n^{-2/3}$ can be easily handled by Markov's inequality. We get that,

$$\mathbb{P}[Z > 0] = \mathbb{P}[Z \geq 1] \leq \frac{\mathbb{E}[Z]}{1}.$$

Since $\mathbb{E}[Z] \rightarrow 0$ as $n \rightarrow \infty$ when $p \ll n^{-2/3}$, we get that $\mathbb{P}[G \text{ contains a copy of } K_4] \rightarrow 0$.

When $p \gg n^{-2/3}$, we want to show that $\mathbb{P}[Z > 0] \rightarrow 1$, i.e., $\mathbb{P}[Z = 0] \rightarrow 0$. We use Chebyshev's inequality to prove this. We first compute the variance of Z .

$$\text{Var}[Z] = \text{Var}\left[\sum_C X_C\right] = \sum_C \text{Var}[X_C] + \sum_{C \neq D} \text{Cov}[X_C, X_D]$$

Since $\mathbb{E}[X_C] = p^6$, we have $\text{Var}[X_C] = p^6 - p^{12}$. Also, for two distinct sets C and D , we consider four different cases depending on the number of vertices they share.

- **Case 1:** $|C \cap D| = 0$. Since no vertex is shared, X_C and X_D are independent and hence $\text{Cov}[X_C, X_D] = 0$.

- **Case 2:** $|C \cap D| = 1$. Since the variables X_C and X_D depend on *pairs* of vertices in the sets C and D , and the two sets do not share any pair, we still have $\text{Cov}[X_C, X_D] = 0$.
- **Case 3:** $|C \cap D| = 2$. Since C and D share a pair of vertices, there are 11 pairs which must all have edges between them in G , for both X_C and X_D to be 1. Thus, we have $\mathbb{E}[X_C X_D] = p^{11}$ and

$$\text{Cov}[X_C, X_D] = \mathbb{E}[X_C X_D] - \mathbb{E}[X_C] \cdot \mathbb{E}[X_D] = p^{11} - p^{12}.$$

- **Case 4:** $|C \cap D| = 3$. in this case C and D share 3 pairs and thus there are 9 distinct pairs of vertices which must all have edges between them for both X_C and X_D to be 1. Thus,

$$\text{Cov}[X_C, X_D] = \mathbb{E}[X_C X_D] - \mathbb{E}[X_C] \cdot \mathbb{E}[X_D] = p^9 - p^{12}.$$

Also, there are $\binom{n}{6} \cdot \binom{6}{4}$ pairs C and D such that $|C \cap D| = 2$, and $\binom{n}{5} \cdot \binom{5}{4}$ pairs such that $|C \cap D| = 3$. Combining the above cases we have,

$$\begin{aligned} \text{Var}[Z] &= \sum_C \text{Var}[X_C] + \sum_{C \neq D} \text{Cov}[X_C, X_D] \\ &= \binom{n}{4} \cdot p^6(1 - p^6) + \binom{n}{6} \cdot \binom{6}{4} \cdot (p^{11} - p^{12}) + \binom{n}{5} \cdot \binom{5}{4} \cdot (p^9 - p^{12}) \\ &= O(n^4 p^6) + O(n^6 p^{11}) + O(n^5 p^9). \end{aligned}$$

Applying Chebyshev's inequality gives

$$\begin{aligned} \mathbb{P}[Z = 0] &\leq \mathbb{P}[|Z - \mathbb{E}[Z]| \geq \mathbb{E}[Z]] \leq \frac{\text{Var}[Z]}{(\mathbb{E}[Z])^2} \\ &= \frac{1}{\binom{n}{4}^2 \cdot p^{12}} \cdot (O(n^4 p^6) + O(n^6 p^{11}) + O(n^5 p^9)) \\ &= O\left(\frac{1}{n^4 p^6}\right) + O\left(\frac{1}{n^2 p}\right) + O\left(\frac{1}{n^3 p^3}\right). \end{aligned}$$

For $p \gg n^{-2/3}$, all the terms on the right tend to 0 as $n \rightarrow \infty$. Hence, $\mathbb{P}[Z = 0] \rightarrow 0$ as $n \rightarrow \infty$. ■

The above analysis can be extended to any graph H of a fixed size. Let Z_H be the number of copies of H in a random graph G generated according to $G_{n,p}$. Using the previous analysis, we have $\mathbb{E}[Z_H] = \Theta\left(n^{|V(H)|} \cdot p^{|E(H)|}\right)$. Hence, $\mathbb{E}[Z] \rightarrow 0$ when $p \ll n^{-|V(H)|/|E(H)|}$ and $\mathbb{E}[Z] \rightarrow \infty$ when $p \gg n^{-|V(H)|/|E(H)|}$. Thus, it might be tempting to conclude that $p = n^{-|V(H)|/|E(H)|}$ is the threshold probability for finding a copy of H . However, con-

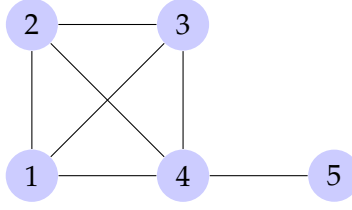


Figure 2: Subgraph H containing K_4

sider the graph in Figure 2. For this graph, we have $|V(H)|/|E(H)| = 5/7$. But for p such that $p \gg n^{-5/7}$ and $p \ll n^{-2/3}$, a random G is extremely unlikely to contain a copy of K_4 and thus also extremely unlikely to contain a copy of H . For an arbitrary graph H , it was shown by Bollobás [Bol81] that the threshold probability is $n^{-\lambda}$, where

$$\lambda = \min_{H' \subseteq H} \frac{|V(H')|}{|E(H')|}.$$

2 Chernoff/Hoeffding Bounds

We now derive sharper concentration bounds for sums of independent random variables. We start by considering n independent Boolean random variables X_1, \dots, X_n , where X_i takes value 1 with probability p_i and 0 otherwise. Let $Z = \sum_{i=1}^n X_i$. We set $\mu = \mathbb{E}[Z] = \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n p_i$. We will try to derive a bound on the probability $\mathbb{P}[Z \geq t]$ for $t = (1 + \delta)\mu$. Using the fact that the function e^x is strictly increasing, we get that for $\lambda > 0$

$$\mathbb{P}[Z \geq (1 + \delta)\mu] = \mathbb{P}[e^{\lambda Z} \geq e^{\lambda(1+\delta)\mu}] \stackrel{\text{(Markov)}}{\leq} \frac{\mathbb{E}[e^{\lambda Z}]}{e^{\lambda(1+\delta)\mu}}.$$

We now have:

$$\begin{aligned} \mathbb{E}[e^{\lambda Z}] &= \mathbb{E}[e^{\lambda(X_1 + \dots + X_n)}] = \mathbb{E}\left[\prod_{i=1}^n e^{\lambda X_i}\right] \stackrel{\text{(independence)}}{=} \prod_{i=1}^n \mathbb{E}[e^{\lambda X_i}] \\ &= \prod_{i=1}^n [p_i e^{\lambda} + (1 - p_i)] \\ &= \prod_{i=1}^n [1 + p_i(e^{\lambda} - 1)]. \end{aligned}$$

At this point, we utilize the simple but very useful inequality:

$$\forall x \in \mathbb{R}, \quad 1 + x \leq e^x.$$

Since all the quantities in the previous calculation are non-negative, we can plug the above inequality in the previous calculation and we get:

$$\mathbb{E} \left[e^{\lambda Z} \right] \leq \prod_{i=1}^n \exp \left((e^\lambda - 1) p_i \right) = \exp \left((e^\lambda - 1) \mu \right)$$

Thus, we get

$$\mathbb{P} [Z \geq (1 + \delta) \mu] \leq \exp \left((e^\lambda - 1) \mu - \lambda (1 + \delta) \mu \right).$$

We now want to minimize the right hand-side of the above inequality, with respect to λ . Setting the derivative of the exponent to zero, we get

$$e^\lambda \mu - (1 + \delta) \mu = 0 \quad \Rightarrow \quad \lambda = \ln(1 + \delta).$$

Using this value for λ , we get

$$\mathbb{P} [Z \geq (1 + \delta) \mu] \leq \frac{\exp \left((e^\lambda - 1) \mu \right)}{\exp \left(\lambda (1 + \delta) \mu \right)} = \frac{e^{\delta \mu}}{(1 + \delta)^{(1 + \delta) \mu}} = \left(\frac{e^\delta}{(1 + \delta)^{1 + \delta}} \right)^\mu.$$

Similarly, we can get that

$$\mathbb{P} [Z \leq (1 - \delta) \mu] \leq \left(\frac{e^{-\delta}}{(1 - \delta)^{1 - \delta}} \right)^\mu.$$

(Note that $\mathbb{P} [Z \leq (1 - \delta) \mu] = \mathbb{P} [e^{-\lambda Z} \geq e^{-\lambda(1 - \delta) \mu}]$.)

When $\delta \in (0, 1)$, the bounds above expressions can be simplified further. It is easy to check that

$$\left(\frac{e^\delta}{(1 + \delta)^{1 + \delta}} \right)^\mu \leq e^{-\delta^2 \mu / 3}, \quad 0 < \delta < 1.$$

So we get:

$$\mathbb{P} [Z \geq (1 + \delta) \mu] \leq e^{-\delta^2 \mu / 3}, \quad \text{for } 0 < \delta < 1.$$

Similarly:

$$\mathbb{P} [Z \leq (1 - \delta) \mu] \leq e^{-\delta^2 \mu / 3}, \quad \text{for } 0 < \delta < 1.$$

Combining the two we get

$$\mathbb{P} [|Z - \mu| \geq \delta \mu] \leq 2 \cdot e^{-\delta^2 \mu / 3}, \quad \text{for } 0 < \delta < 1.$$

2.1 Coin tosses once more

We will now compare the above bound with what we can get from Chebyshev's inequality. Let's assume that X_1, \dots, X_n are independent coin tosses, with $\mathbb{P}[X_i = 1] = \frac{1}{2}$. We want to get a bound on the value of $Z = \sum_{i=1}^n X_i$. Using Chebyshev's inequality, we get that

$$\mathbb{P}[|Z - \mu| \geq \delta\mu] \leq \frac{\text{Var}[Z]}{\delta^2\mu^2}.$$

And since in this particular case we have that $\text{Var}[Z] = n/4$ and $\mu = n/2$, we get that

$$\mathbb{P}[|Z - \mu| \geq \delta\mu] \leq \frac{1}{\delta^2 n}.$$

The above bound is only inversely polynomial in n , while the Chernoff-Hoeffding bound gives

$$\mathbb{P}[|Z - \mu| \geq \delta\mu] \leq 2 \cdot \exp(-\delta^2 n/6),$$

which is exponentially small in n . This fact will prove very useful when taking a union bound over a large collection of events, each of which may be bounded using a Chernoff-Hoeffding bound. For example, consider the case where for m sets $S_1, \dots, S_m \subseteq [n]$, we define

$$Z_{S_i} = \sum_{j \in S_i} X_j.$$

While the variables Z_{S_1}, \dots, Z_{S_m} are *not* necessarily independent, each of these is a sum of few X_j variables, which are independent. Thus, we can say that for any S_i ,

$$\mathbb{P}\left[\left|Z_{S_i} - \frac{|S_i|}{2}\right| \geq t\right] \leq 2 \cdot \exp(-2t^2/(3|S_i|)) \leq 2 \cdot \exp(-2t^2/(3n)),$$

where we choose $\delta = 2t/|S_i|$ so that $\delta|S_i|/2 = t$. Thus, by a union bound over all $i \in [m]$, we get that

$$\mathbb{P}\left[\exists i \in [m]. \left|Z_{S_i} - \frac{|S_i|}{2}\right| \geq t\right] \leq 2m \cdot \exp(-2t^2/(3n)).$$

Thus, when $t = \sqrt{3n \cdot \ln m}$, the probability of the above event is at most $2/m$. Check that it just using Chebyshev's inequality does not allow for such a strong bound on the probability of the above event.

References

- [Bol81] Béla Bollobás, *Threshold functions for small subgraphs*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 90, Cambridge Univ Press, 1981, pp. 197–206. [4](#)

[ER60] Paul Erdős and A Rényi, *On the evolution of random graphs*, Publ. Math. Inst. Hungary. Acad. Sci 5 (1960), 17–61. [1](#)