Mathematical Toolkit **Autumn 2018**

Homework 4

Due: December 7, 2018

Note: *You may discuss these problems in groups. However, you must write up your own solutions and mention the names of the people in your group. Also, please do mention any books, papers or other sources you refer to. It is recommended that you typeset your solutions in LaTeX.*

1. **Gaussian Random Variables.** **[5+5+5]**

   Prove the following very useful facts about Gaussian random variables:

   (a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be two vectors. Let $\mathbf{g} \in \mathbb{R}^n$ be a random vector such that each coordinate g_i of $\mathbf{g}$ is distributed as a Gaussian random variable with mean 0 and variance 1, and any two coordinates g_i, g_j (for $i \neq j$) are independent. Then show that

   $$\mathop{\mathbb{E}}_{\mathbf{g}}\left[\langle \mathbf{u}, \mathbf{g} \rangle \cdot \langle \mathbf{v}, \mathbf{g} \rangle\right] \;=\; \langle \mathbf{u}, \mathbf{v} \rangle \;.$$

   (b) Let g be a Gaussian random variable with mean 0 and variance 1. Show that for any $t \in \mathbb{R}$, we have

   $$\mathbb{E}\left[e^{tg}\right] \;=\; e^{t^2/2}\,.$$

   Comparing coefficients of t^{2k} on both sides, use this to show that for any $k \in \mathbb{N}$,

   $$\mathbb{E}\left[g^{2k}\right] \;=\; \frac{(2k)!}{2^k \cdot k!}\,.$$

   (c) Let g_1, g_2, g_3 and g_4 be (not necessarily independent) Gaussian random variables with mean 0. Additionally, assume that for *all* coefficients $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$, the linear combination $\alpha_1 g_1 + \cdots + \alpha_4 g_4$ is also a Gaussian random variable (note that this is not always true if $g_1, \ldots, g_4$ are not independent, but here we are restricting ourselves to $g_1, \ldots, g_4$ which satisfy this assumption).

   Consider the function $\mathbb{E}_{g_1,g_2,g_3,g_4}\left[e^{t_1 g_1 + t_2 g_2 + t_3 g_3 + t_4 g_4}\right]$ in the variables t_1, t_2, t_3, t_4 and use it to show that

   $$\mathbb{E}\left[g_1 g_2 g_3 g_4\right] \;=\; \mathbb{E}\left[g_1 g_2\right] \cdot \mathbb{E}\left[g_3 g_4\right] + \mathbb{E}\left[g_1 g_3\right] \cdot \mathbb{E}\left[g_2 g_4\right] + \mathbb{E}\left[g_1 g_4\right] \cdot \mathbb{E}\left[g_2 g_3\right] \;.$$

   This shows that for *any* four Gaussian random variables, the expectation of their product can be expressed in terms of their pairwise correlations! This is a special case of what is known as Wick's theorem, which can also be proved by the above method.

2. **Supremum of Gaussians.** **[5+5]**

(a) Let $g \sim N(0,1)$ be a Gaussian random variable with mean 0 and variance 1. Show that for $t \geq 1$

$$\mathbb{P}\left[g \geq t\right] \quad = \quad \int_t^\infty \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}\, dx \quad \leq \quad e^{-t^2/2}\,.$$

(b) Let $g_1, \ldots, g_n \sim N(0,1)$ be independent Gaussian random variables. Show that

$$\mathbb{E}\left[\max_{i \in [n]} |g_i|\right] \quad \leq \quad 4\sqrt{\ln n}\,.$$

You may use the fact that for a non-negative random variable Z, the expectation can be computed as $\mathbb{E}\left[Z\right] = \int_0^\infty \mathbb{P}\left[Z \geq t\right] dt$.