Mathematical Toolkit **Autumn 2016**

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1 Adjoint of a linear transformation

Definition 1.1 *Let V, W be inner product spaces over the same field $\mathbb{F}$ and let $\varphi : V \to W$ be a linear transformation. A transformation $\varphi^* : W \to V$ is called an* adjoint *of φ if*

$$\langle \varphi(v), w \rangle = \langle v, \varphi^*(w) \rangle \quad \forall v \in V, w \in W.$$

Example 1.2 *Let $V = W = \mathbb{C}^n$ with the inner product $\langle u, v \rangle = \sum_{i=1}^{n} u_i \cdot \overline{v_i}$. Let $\varphi : V \to V$ be represented by the matrix A. Then φ^* is represented by the matrix $\overline{A^T}$.*

Exercise 1.3 *Let $\varphi_{\text{left}} : Fib \to Fib$ be the left shift operator as before, and let $\langle f, g \rangle$ for $f, g \in Fib$ be defined as $\langle f, g \rangle = \sum_{n=0}^{\infty} \frac{f(n)g(n)}{C^n}$ for $C > 4$. Find φ^*_{left}.*

We will prove that every linear transformation has a unique adjoint. However, we first need the following characterization of linear transformations from V to $\mathbb{F}$.

Proposition 1.4 (Riesz Representation Theorem) *Let V be a finite-dimensional inner product space over $\mathbb{F}$ and let $\alpha : V \to \mathbb{F}$ be a linear transformation. Then there exists a unique $z \in V$ such that $\alpha(v) = \langle v, z \rangle \ \forall v \in V$.*

We only prove the theorem here for finite-dimensional spaces. However, the theorem holds for any Hilbert space.

Proof: Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then check that

$$z = \sum_{i=1}^{n} \overline{\alpha(w_i)} \cdot w_i$$

must be the unique z satisfying the required property. ■

This can be used to prove the following:

Proposition 1.5 *Let V, W be finite dimensional inner product spaces and let $\varphi : V \to W$ be a linear transformation. Then there exists a unique $\varphi^* : W \to V$, such that*

$$\langle \varphi(v), w \rangle \;=\; \langle v, \varphi^*(w) \rangle \quad \forall v \in V, w \in W\,.$$

Proof: For each $w \in W$, the map $\langle \varphi(\cdot), w \rangle : V \to \mathbb{F}$ is a linear transformation and hence there exists a unique $z_w \in V$ satisfying $\langle \varphi(v), w \rangle = \langle v, z_w \rangle \;\forall v \in V$. Consider the map $\alpha : W \to V$ defined as $\alpha(w) = z_w$. By definition of α,

$$\langle \varphi(v), w \rangle \;=\; \langle v, \alpha(w) \rangle \quad \forall v \in V, w \in W\,.$$

To check that α is linear, we note that $\forall v \in V,\; \forall w_1, w_2 \in W$,

$$\langle v, \alpha(w_1 + w_2) \rangle \;=\; \langle \varphi(v), w_1 + w_2 \rangle \;=\; \langle \varphi(v), w_1 \rangle + \langle \varphi(v), w_2 \rangle \;=\; \langle v, \alpha(w_1) \rangle + \langle v, \alpha(w_2) \rangle\,,$$

which implies $\alpha(w_1 + w_2) = \alpha(w_1) + \alpha(w_2)$. $\alpha(c \cdot w) = c \cdot \alpha(w)$ follows similarly. ∎

Note that the above proof only requires the Riesz representation theorem (to define z_w) and hence also works for Hilbert spaces.

Definition 1.6 *A linear transformation $\varphi : V \to V$ is called* self-adjoint *if $\varphi = \varphi^*$. Linear transformations from a vector space to itself are called* linear operators.

Example 1.7 *The transformation represented by matrix $A \in \mathbb{C}^{n \times n}$ is self-adjoint if $A = \overline{A^T}$. Such matrices are called Hermitian matrices.*

Proposition 1.8 *Let V be an inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then*

- *All eigenvalues of φ are real.*
- *If $\{w_1, \ldots, w_n\}$ are eigenvectors corresposnding to distinct eigenvalues then they are mutually orthogonal.*

We will use the following statement (which we'll prove later) to prove the spectral theorem.

Proposition 1.9 *Let V be a finite-dimensional inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then φ has at least one eigenvalue.*

Using the above proposition, we will prove the spectral theorem below for finite dimensional vector spaces. The proof below can also be made to work for Hilbert spaces (using the axiom of choice). The above proposition, which gives the existence of an eigenvalue is often proved differently for finite and infinite-dimensional spaces, and the proof for infinite-dimensional Hilbert spaces requires additional conditions on the operator φ. We first prove the spectral theorem assuming the above proposition.

Proposition 1.10 (Real spectral theorem) *Let V be a finite-dimensional inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then φ is orthogonally diagonalizable.*

Proof: By induction on the dimension of V. Let $\dim(V) = 1$. Then by the previous proposition φ has at least one eigenvalue, and hence at least one eigenvector, say w. Then $w / \|w\|$ is a unit vector which forms a basis for V.

Let $\dim(V) = k+1$. Again, by the previous proposition φ has at least one eigenvector, say w. Let $W = \text{Span}\left(\{w\}\right)$ and let $W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0\}$. Check the following:

- $W^{\perp}$ is a subspace of V.
- $\dim(W^{\perp}) = k$.
- $W^{\perp}$ is invariant under φ i.e., $\forall v \in W^{\perp}$, $\varphi(v) \in W^{\perp}$.

Thus, we can consider the operator $\varphi' : W^{\perp} \to W^{\perp}$ defined as

$$\varphi'(v) \ := \ \varphi(v) \qquad \forall v \in W^{\perp}.$$

Then, φ' is a self-adjoint (check!) operator defined on the k-dimensional space $W^{\perp}$. By the induction hypothesis, there exists an orthonormal basis $\{w_1, \ldots, w_k\}$ for $W^{\perp}$ such that each w_i is an eigenvector of φ. Thus $\left\{w_1, \ldots, w_k, \frac{w}{\|w\|}\right\}$ is an orthonormal basis for V, comprising of eigenvectors of φ. ■

We will prove the proposition about existence of an eigenvalue in the next lecture.