Mathematical Toolkit **Autumn 2016**

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1 Linear Transformations

Definition 1.1 *Let V and W be vector spaces over the same field $\mathbb{F}$. A map $\varphi : V \to W$ is called a* linear transformation *if*

- $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2) \quad \forall v_1, v_2 \in V$.
- $\varphi(c \cdot v) = c \cdot \varphi(v) \quad \forall v \in V$.

Example 1.2 *The following are all linear transformations:*

- *A matrix $A \in \mathbb{R}^{m \times n}$ defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.*
- $\varphi : C([0,1], \mathbb{R}) \to C([0,2], \mathbb{R})$ *defined by* $\varphi(f)(x) = f(x/2)$.
- $\varphi : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ *defined by* $\varphi(f)(x) = f(x^2)$.
- $\varphi : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ *defined by* $\varphi(f)(x) = f(1-x)$.
- $\varphi_{\text{left}} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ *defined by* $\varphi_{\text{left}}(f)(n) = f(n+1)$.
- *The derivative operator acting on $\mathbb{R}[x]$.*

Proposition 1.3 *Let V, W be vector spaces over $\mathbb{F}$ and let B be a basis for V. Let $\alpha : B \to W$ be an arbitrary map. Then there exists a unique linear transformation $\varphi : V \to W$ satisfying $\varphi(v) = \alpha(v)\ \forall v \in B$.*

Definition 1.4 *Let $\varphi : V \to W$ be a linear transformation. We define its* kernel *and* image *as:*

- $\ker(\varphi) := \{v \in V \mid \varphi(v) = 0_W\}$.
- $\operatorname{im}(\varphi) = \{\varphi(v) \mid v \in V\}$.

Proposition 1.5 $\ker(\varphi)$ *is a subspace of V and* $\operatorname{im}(\varphi)$ *is a subspace of W.*

Proposition 1.6 (rank-nullity theorem) *If V is a finite dimesional vector space and $\varphi : V \to W$ is a linear transformation, then*

$$\dim(\ker(\varphi)) + \dim(\mathrm{im}(\varphi)) \;=\; \dim(V)\,.$$

$\dim(\mathrm{im}(\varphi))$ is called the rank and $\dim(\ker(\varphi))$ is called the nullity of φ.

Example 1.7 *Consider the matrix A which defines a linear transformation from $\mathbb{F}_2^7$ to $\mathbb{F}_2^3$:*

$$A \;=\; \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

- $\dim(\mathrm{im}(\varphi)) = 3$.
- $\dim(\ker(\varphi)) = 4$.
- *Check that* $\ker(\varphi)$ *is a* code *which can recover from one bit of error.*
- *Check that this is also true for the $(2^k - 1) \times k$ matrix A_k where the i^{th} column is the number i written in binary (with the most significant bit at the top).*

This code is known as the Hamming Code and the matrix A is called the parity-check matrix of the code.

2 Eigenvalues and eigenvectors

Definition 2.1 *Let V be a vector space over the field $\mathbb{F}$ and let $\varphi : V \to V$ be a linear transformation. $\lambda \in \mathbb{F}$ is said to be an* eigenvalue *of φ if there exists $v \in V \setminus \{0_V\}$ such that $\varphi(v) = \lambda \cdot v$. Such a vector v is called an* eigenvector *corresponding to the eigenvalue λ. The set of eigenvalues of φ is called its* spectrum*:*

$$\mathrm{spec}(\varphi) = \{\lambda \mid \lambda \textit{ is an eigenvalue of } \varphi\}\ .$$

Example 2.2 *Consider the following transformations:*

- *Differentiation is a linear transformation on the class of infinitely differentiable functions and each function of the form $c \cdot \exp(\lambda x)$ is an eigenvector with eigenvalue λ.*
- *Consider the transformation $\varphi_{\mathrm{left}} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$. Any geometric progression with common ratio r is an eigenvector of φ_{left} with eigenvalue r (and these are the only eigenvectors for this transformation).*

Proposition 2.3 *Let* $U_\lambda = \{v \in V \mid \varphi(v) = \lambda \cdot v\}$. *Then for each* $\lambda \in \mathbb{F}$, U_λ *is a subspace of* V.

Note that $U_\lambda = \{0_V\}$ if λ is not an eigenvalue. The dimension of this subspace is called the geometric multiplicity of the eigenvalue λ.

Proposition 2.4 *Let* $\lambda_1, \ldots, \lambda_k$ *be distinct eigenvalues of* φ *with associated eigenvectors* $v_1, \ldots, v_k$. *Then the set* $\{v_1, \ldots, v_k\}$ *is linearly independent.*

Definition 2.5 *A transformation* $\varphi : V \to V$ *is said to be* diagonalizable *if there exists a basis of* V *comprising of eigenvectors of* φ.

Exercise 2.6 *Recall that* $\text{Fib} = \left\{f \in \mathbb{R}^{\mathbb{N}} \mid f(n) = f(n-1) + f(n-2)\ \forall n \geq 2\right\}$. *Show that* $\varphi_{\text{left}} : \text{Fib} \to \text{Fib}$ *is diagonalizable. Express the sequence by* $f(0) = 1, f(1) = 1$ *and* $f(n) = f(n-1) + f(n-2)\ \forall n \geq 2$ *(known as Fibonacci numbers) as a linear combination of eigenvectors of* φ_{left}.