Mathematical Toolkit	**Autumn 2016**

Homework 1

Due: October 20, 2016

Note: *You may discuss these problems in groups. However, you must write up your own solutions and mention the names of the people in your group. Also, please do mention any books, papers or other sources you refer to. It is recommended that you typeset your solutions in LaTeX.*

1. Let $A \in \mathbb{F}_2^{m \times n}$ be a matrix with entries in the field $\mathbb{F}_2$ and let $m < n$. Let all rows of A be linearly independent in the vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$.

   (a) What is the dimension of the space $\ker(A)$?

   (b) How many vectors $x \in \mathbb{F}_2^n$ satisfy the system of equations $Ax = 0$? (Note that here 0 denotes the zero vector in $\mathbb{F}_2^m$.)

   (c) Let $b \in \mathbb{F}_2^m$ be such that the system of equations $Ax = b$ has at least one solution, say x_0. Show that $\{x - x_0 \mid Ax = b\} = \ker(A)$. What is the total number of solutions to the system $Ax = b$?

   For this problem you may use the fact that for a matrix $A \in \mathbb{F}^{m \times n}$ for any field $\mathbb{F}$, if $R \subseteq \mathbb{F}^n$ denotes the set of its rows and $C \subseteq \mathbb{F}^m$ denotes the set of its columns, then

   $$\dim(\mathrm{Span}\,(R)) \;=\; \dim(\mathrm{Span}\,(C))\,.$$

   The quantity $\dim(\mathrm{Span}\,(R))$ is called the row-rank of A and $\dim(\mathrm{Span}\,(C))$ is called the column-rank of A.

2. Let V be a vector space over the field $\mathbb{F}$ (not necessarily $\mathbb{R}$ or $\mathbb{C}$) and let $f : V \to [0,1]$ be a function satisfying $f(c \cdot v + d \cdot w) \geq \min\{f(v), f(w)\}$ for all $c, d \in \mathbb{F}$ and all $v, w \in V$. Show that

   (a) $f(0_V) \;\geq\; f(v)$ for all $v \in V$.

   (b) For any $t \in [0, f(0_V)]$, the space $V_t = \{v \in V \mid f(v) \geq t\}$ is a subspace of V.

3. Let $\mathbb{F}$ be a field and let $P(x) = x^2 + bx + c \in \mathbb{F}[x]$ be a polynomial which has two distinct *nonzero* roots r_1 and r_2 in $\mathbb{F}$. Let $\varphi_1 : \mathbb{F}^3 \to \mathbb{F}$ be defined as $\varphi(x_1, x_2, x_3) = x_1 + bx_2 + cx_3$. Let $\varphi_2 : \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ be defined by $(\varphi_2(f))(n) = \varphi_1(f(n), f(n+1), f(n+2))$ for each $f \in \mathbb{R}^{\mathbb{N}}$ and each $n \in \mathbb{N}$. Show that $\dim(\ker(\varphi_2)) \;\geq\; 2$.

4. Consider the vector space $\mathbb{R}[x]$ of polynomials in a single variable x with coefficients in $\mathbb{R}$. Define the function $\mu : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}$ as

$$\mu(P, Q) \; = \; \text{degree}(P \cdot Q) \quad \text{for all } P, Q \in \mathbb{R}[x]\,,$$

where $P \cdot Q$ denotes the product of the two polynomials P and Q (which is another polynomial). Is the function μ an inner product? Justify your answer.

5. Let V be a finite dimensional vector space over a field $\mathbb{F}$ and $\alpha, \beta : V \to V$ be linear operators. Show that for every $\lambda \in \mathbb{F}$ (including $0_{\mathbb{F}}$), λ is an eigenvalue of $\alpha\beta$ if and only if λ is an eigenvalue of $\beta\alpha$.

6. A linear operator $\varphi : V \to V$ is called a projection if $\varphi^2 = \varphi$ i.e., $\varphi^2(v) = \varphi(v)\ \forall v \in V$.

   (a) If V is a vector space over a field $\mathbb{F}$, $\varphi : V \to V$ is a projection, and $\lambda \in \mathbb{F}$ is an eigenvalue of φ, then show that $\lambda \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$.

   (b) If V is an inner product space over $\mathbb{F}$ (taken to be $\mathbb{R}$ or $\mathbb{C}$ for this part) and φ is a projection, is it always true that $\varphi = \varphi^*$? Justify your answer.

7. Let V, W be finite-dimensional inner product spaces $\varphi : V \to W$ and let $\varphi^* : W \to V$ denote the adjoint of φ. Prove the following:

   (a) $(\varphi^*)^* = \varphi$.

   (b) $\ker(\varphi) = (\text{im}(\varphi^*))^{\perp}$.

   (c) $\text{im}(\varphi) = (\ker(\varphi^*))^{\perp}$.

   (d) $\text{rank}(\varphi) = \text{rank}(\varphi^*)$.

   (e) If $A \in \mathbb{C}^{m \times n}$ is a matrix, then show that row-rank of A equals the column-rank of A.

   Note that the above only proves row-rank equals column-rank when A is a matrix with entries in $\mathbb{C}$ (or $\mathbb{R}$). There is actually a much simpler proof which proves this over all fields! We will discuss this in class later.