TTIC/CMSC 31150 Mathematical Toolkit **Spring 2013**

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Recall that G is a d-regular graph, and A_G denotes its adjacency matrix with $d = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq -d$ its n eigenvalues. Laplacian of G is denoted as $L_G = dI - A_G$. Normalized laplacian, $N_G = I - 1/dA_G$ with $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2$ its n eigenvalues.

1 Saving Random bits Continued

Recall that we were trying to prove the following lemma.

Lemma 1.1 *If $\mu(G) \leq 9/10 \cdot d$ and $S \subseteq V$ with $|S| \geq n/2$ then*

$$\mathbb{P}\left[\textit{Random walk of length } l \textit{ avoids } S\right] = 2^{-\Omega(l)}$$

Proof: Recall that the number of random walk of length l can be represented as $\mathbf{1}^T A^l \mathbf{1} = d^l \cdot n$. Similarly, the number of walks of length l that avoid S is $\mathbf{1}^T \overline{A}^l \mathbf{1}$ where

$$\overline{A}_{ij} = \begin{cases} 0 & \text{if } i \in S \text{ or } j \in S \\ A_{ij} & \text{otherwise} \end{cases}$$

Last class, we proved that

$$\text{all eigenvalues of } |\overline{A}| \leq \left(1 - \frac{|S|}{n}\right) \cdot d + \frac{|S|}{n} \cdot \mu$$

Therefore we can give a following upper bound.

$$\begin{aligned} \text{Number of walks avoiding } S &= \mathbf{1}^T \overline{A}^l \mathbf{1} \\ &\leq \left(\frac{\mu + d}{2}\right)^l (\mathbf{1}^T \mathbf{1}) \\ &\leq \left(\frac{19d}{20}\right)^l (\mathbf{1}^T \mathbf{1}) \\ &= \left(\frac{19d}{20}\right)^l \cdot n \end{aligned}$$

$$\begin{aligned} \mathbb{P}\left[\text{walk avoids } S\right] &\leq \frac{\left(\frac{19}{20}\right)^l \cdot d^l \cdot n}{d^l \cdot n} \\ &= 2^{-\Omega(n)} \end{aligned}$$

■

2 Explicit Construction of an Expander

2.1 Expanders as an approximation of K_n

First, consider eigenvalues of A_{K_n}. Observe that $A_{K_n} = J - I_n$. Since the eigenvalue of J is n and 0 and the eigenvalue of I_n is all 1,

- Eigenvalues of A_{K_n} : $n-1, -1, -1, \ldots, -1$
- Eigenvalues of N_{K_n} : $0, 1+\frac{1}{n-1}, 1+\frac{1}{n-1}, \ldots, 1+\frac{1}{n-1}$

An expander G approximates K_n in a following sense.

$$(1-\varepsilon)N_{K_n} \preceq N_G \preceq (1+\varepsilon)N_{K_n}$$

Definition 2.1 *$A \preceq B$ if $B-A$ is a positive semi-definite matrix. That is, $\forall \mathbf{z} \in \mathbb{R}^n$, $\mathbf{z}^T A \mathbf{z} \leq \mathbf{z}^T B \mathbf{z}$*

2.2 Squaring a graph

Here, we explain a procedure that will increase the expansion of a graph.

Definition 2.2 (Squaring a graph) *Consider $G = (V, E)$. Also assume that G does not contain any 4-cycle. Then define $G^2 = (V', E')$ as following. $V' = V$ and $(i, j) \in E'$ if there exists a length 2 path from i to j where $i \neq j$.*

The adjacency matrix of G^2 can be then denoted as $A^2 - dI$. Therefore the eigenvalues of this graph are $\mu_1^2 - d = d^2 - d$, $\mu_2^2 - d, \ldots, \mu_n^2 - d$, not necessarily in order. Then we can make two observations.

1. $\mu_2(G^2) \leq \mu(G)^2 - d$
2. $\mu(G^2) \leq \max(\mu^2 - d, d)$

Similarly, we can analyze the eigenvalues of normalized laplacian of G^2.

- Eigenvalues of N_G : $0 = 1 - \frac{\mu_1}{d}, \ldots, 1 - \frac{\mu_n}{d}$
- Eigenvalues of N_{G^2} : $0 = 1 - \frac{\mu_1^2 - d}{d^2 - d}, \ldots, 1 - \frac{\mu_n^2 - d}{d^2 - d}$

Then we want to relate second eigenvalue of G with G^2.

$$\begin{aligned}
\lambda_2(G^2) &= \min\left(1 - \frac{\mu_2^2 - d}{d^2 - d}, 1 - \frac{\mu_n^2 - d}{d^2 - d}\right) \\
&\geq 1 - \frac{\mu^2 - d}{d^2 - d} \\
&\geq 1 - \frac{\mu^2}{d^2}
\end{aligned}$$

Since $\mu/d \le 1 - \lambda_2(G)$,

$$\lambda_2(G^2) \;\ge\; 1 - (1 - \lambda_2(G))^2$$

Then finally, we get

$$1 - \lambda_2(G^2) \le (1 - \lambda_2(G))^2$$

2.3 Line graph

Definition 2.3 *H is a line graph of G if*

- $V(H) = E(G)$
- $(e_1, e_2) \in E(H)$ *if e_1 and e_2 share a vertex in G*

Under this definition, we can make following observations.

Fact 2.4 *For a line graph H of d-regular graph G, the following holds.*

- $|V(H)| = |E(G)| = nd/2$
- *$deg(v) = 2d - 2$ for any $v \in V(H)$.*
- *Each v is incident on exactly two d-cliques*

Now we want to compute the eigenvalues of the line graph.

Lemma 2.5 *All eigenvalues of L_G is eigenvalues of L_H with the remaining eigenvalues being $2d$.*

Proof: First recall the following fact about laplacians.

$$\mathbf{x}^T L_G \mathbf{x} = \sum_{(i,j)\in E} |x_i - x_j|^2$$

Then this immediately implies the following decomposition of a laplacian.

$$\begin{aligned} L_G &= \sum_{(i,j)\in E} (\mathbf{1}_i - \mathbf{1}_j)(\mathbf{1}_i - \mathbf{1}_j)^T \\ &= UU^T \end{aligned}$$

where $\mathbf{1}_i$ denotes a vector with 1 on ith position and 0 on other positions, and U is a $n \times m$ matrix with each column vector $u_e = \mathbf{1}_i - \mathbf{1}_j$ with $e = (i, j)$.
Consider $|U|$ where each $U_{ij} = |U_{ij}|$. Then since $L_G = dI_n - A_G = UU^T$, $|U|\,|U|^T = dI_n + A_G$

Claim 2.6 $|U|\,|U|^T = 2I_m + A_H$

Proof of claim: Consider $\left(|U|\,|U|^T\right)_{e_1e_2}$

$$\left(|U|\,|U|^T\right)_{e_1e_2} = \begin{cases} 2 & \text{if } e_1 = e_2 \\ 1 & \text{if } e_1 \text{ and } e_2 \text{ share a vertex} \\ 0 & \text{otherwise} \end{cases}$$

□

Claim 2.7 $|U|^T\,|U|$ *share the same eigenvalues with* $|U|\,|U|^T$

Proof of claim: Consider a singular value decomposition of $|U| = \sum \sigma_i u_i v_i^T$ Then

$$\begin{aligned} |U|\,|U|^T &= \left(\sum \sigma_i u_i v_i^T\right)\left(\sum \sigma_i u_i v_i^T\right)^T \\ &= \left(\sum \sigma_i u_i v_i^T\right)\left(\sum \sigma_i v_i u_i^T\right) \\ &= \sum \sigma_i^2 u_i u_i^T \end{aligned}$$

Similarly,

$$|U|^T\,|U| = \sum \sigma_i^2 v_i v_i^T$$

Here σ_i^2 are their eigenvalues, and they are all equal. □

Now suppose α is an eigenvalue of L_G

$\Rightarrow$ α is an eigenvalue of $dI_n - A_G$

$\Rightarrow$ $d - \alpha$ is an eigenvalue of A_G

$\Rightarrow$ $2d - \alpha$ is an eigenvalue of $dI_n + A_G = |U|\,|U|^T$

$\Rightarrow$ $2d - \alpha$ is an eigenvalue of $|U|^T\,|U| = 2I_m + A_H$

$\Rightarrow$ $2d - 2 - \alpha$ is an eigenvalue of A_H

$\Rightarrow$ α is an eigenvalue of L_H

Here the remaining eigenvalue to $|U|^T\,|U|$ should be 0. Therefore remaining eigenvalues to L_H should be $2d$. ■

Therefore, for the line graphs we have the following eigenvalues.

- Eigenvalues for L_H : $0 = d - \mu_1 \leq \ldots \leq d - \mu_n = 2d \ldots 2d$
- Eigenvalues for A_H : $2(d-1) = d - 2 + \mu_1 \geq d - 2 + \mu_2 \geq \ldots \geq -2 \ldots -2$

2.4 Putting things together

Definition 2.8 (Line Graph Product) *Let G be a d-regular graph with $|V| = n$. Let Z be a k-regular expander on d-vertices. Then define $G \textcircled{L} Z$ as a graph obtained by replacing every d-clique in line graph of G by a copy of Z.*

Following is the summary of graph quantities throughout our procedure. First assume that $\mu(G) \leq 4d/5$ or $\lambda_2(G) \geq 1/5$, and select k so that $d \cong 16k^4$ while $\mu(Z) \leq \varepsilon k$. Then our algorithm repeats the following process from G to $\left((G \textcircled{L} Z)^2\right)^2$.

	G	H	$G \textcircled{L} Z$	$\left((G \textcircled{L} Z)^2\right)^2$
λ_2	$1/5$	$1/10$	$\cong 1/10$	$\geq 1/5$
V	n	$nd/2$	$nd/2$	$nd/2$
degree	d	$2(d-2)$	$2k$	d

We showed all the parts except λ_2 from H to $G \textcircled{L} Z$.

Lemma 2.9 *Let L_H, N_H denote laplacian and normalized laplacian of H and $L_{G \textcircled{L} Z}$, $N_{G \textcircled{L} Z}$ that of $G \textcircled{L} Z$. Then*

$$(1-\varepsilon)N_H \preceq N_{G \textcircled{L} Z} \preceq (1+\varepsilon)N_H$$

Proof: Consider following decomposition of L_H.

$$\begin{aligned} L_H &= \sum_{(i,j) \in E_H} L_{\{i,j\}} \\ &= \sum_{r \in R} \sum_{(i,j) \in r\text{th clique}} L_{\{i,j\}} \\ &= \sum_{r \in R} L_{H_r} \end{aligned}$$

where E_H denote the set of edges for H, R denotes the index set for cliques in H, H_r denote the actual rth clique.

Similarly,

$$L_{G \textcircled{L} Z} = \sum_{r \in R} L_{Z_r}$$

where Z_r denote the rth copy of Z.

By assumption we have $(1-\varepsilon)N_{K_d} \preceq N_Z \preceq (1+\varepsilon)N_{K_d}$. And it is easy to see that if $B_1 \preceq A_1$ and

$B_2 \preceq A_2$ then $B_1 + B_2 \preceq A_1 + A_2$. Then

$$\begin{aligned}
N_{G \textcircled{L} Z} &= \frac{1}{2k} L_{G \textcircled{L} Z} \\
&= \frac{1}{2k} \sum_{r \in R} L_{Z_r} \\
&= \frac{1}{2} \sum_{r \in R} N_{Z_r} \\
&\succeq \frac{1}{2} \sum_{r \in R} (1-\varepsilon) N_{H_r} \\
&= \frac{1}{2} \sum_{r \in R} \frac{(1-\varepsilon)}{(d-1)} L_{H_r} \\
&= \frac{(1-\varepsilon)}{2(d-1)} \sum_{r \in R} L_{H_r} \\
&= (1-\varepsilon) N_H
\end{aligned}$$

Similarly, $N_{G \textcircled{L} Z} \preceq (1+\varepsilon) N_H$ ■

References

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