REU APPRENTICE PROBLEMS

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1. Puzzle Problems

Problem 1. Prove that the number of people in the world who have made an odd number of handshakes is even.

Problem 2. Consider an 8×8 chessboard, with two opposite corners removed. Is it possible to tile this with non-overlapping dominoes?

Problem 3. We are are given thirteen weights (say integers) with the following property: if any one weight is removed, the remaining twelve can be split into two groups of six each such that the two groups have equal total weight. Prove that all thirteen weights must be equal.

2. Vectors and fields

Problem 4. Let $g(x) = (x - \alpha_1) \cdots (x - \alpha_n)$, where the numbers α_i are distinct real number. Let $f_i(x) = g(x)/(x - \alpha_i)$. Show that the polynomials $f_1, \ldots, f_n$ are linearly independent.

Problem 5. Show that if F is a number field, then $F \supset \mathbb{Q}$.

Problem 6. Prove that:
(1) $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a number field.
(2) $\mathbb{Q}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$ is a number field.

Problem 7. Prove that $1, \sqrt{2}, \sqrt{3}$ are linearly independent over $\mathbb{Q}$.

Problem 8. Prove that the functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots$ are linearly independent. (Hint: consider $\int_0^{2\pi} \sin(kx)\cos(lx)dx$)

Problem 9. If $v_1, \ldots, v_k$ are vectors in $\mathbb{Z}^n$ that are linearly independent over $\mathbb{Q}$, then show that they are also linearly independent over $\mathbb{R}$.

3. Lagrange Interpolation and Polynomials

Problem 10. Your instructor constructed two polynomials $P_1(x)$ and $P_2(x)$, each of degree (at most) d and evaluated them both at points $a_1, \ldots, a_n \in \mathbb{R}$. At this point things got a bit mixed up and he no longer knows which value is coming from which polynomial.

You are given n pairs of values: $(b_1, c_1), \ldots, (b_n, c_n)$. We know that at each point a_i, either $b_i = P_1(a_i)$ and $c_i = P_2(a_i)$, or $b_i = P_2(a_i)$ and $c_i = P_1(a_i)$ (things can be different for each i). Can you figure out both the polynomials $P_1(x)$ and $P_2(x)$? What relation do you need between d and n to solve this problem? Do you need any conditions on the coefficients of the polynomials?

4. Miscellaneous problems

Problem 11. Prove that for an undirected graph in which every vertex has degree at most d, the vertices can be properly colored with $d + 1$ colors.

Problem 12. Find a necessary and sufficient condition for a graph to be colorable by two colors.

Problem 13. Consider a regular n-gon inscribed in a circle of radius 1, with consecutive vertices $P_0, P_1, \ldots, P_{n-1}$. Show that the product of all the lengths $\overline{P_0 P_i}$ is equal to n.

(Hint: polynomials, complex numbers.)

5. The Fibonacci space

Consider the vector space V consisting of sequences $(a_0, a_1, \ldots)$ of real numbers.

Say that $s \in V$ is a *Fibonacci-type* sequence if for every $n \in \mathbb{N}$, we have $s_{n+2} = s_{n+1} + s_n$. For example, the Fibonacci numbers form a Fibonacci-type sequence.

Problem 14. (a) The set $\underline{\text{Fib}}$, consisting of all the Fibonacci-type sequences, is a subspace of V with dimension 2. Show that the sequences s and t form a basis of $\underline{\text{Fib}}$, where $s_0 = 1$, $s_1 = 0$, $t_0 = 0$ and $t_1 = 1$.

(b) Find a basis of $\underline{\text{Fib}}$ consisting of two geometric progressions: $(1, r, r^2, \ldots)$ and $(1, s, s^2, \ldots)$. Then if F_n is the n-th Fibonacci number, we will have $F_n = \alpha r^n + \beta s^n$. Find r, s, α, β.

6. Matrix rank

Problem 15. Elementary row and column operations do not change either the row-rank or the column-rank of the matrix. Prove this by proving that

(a) elementary column operations do not change the column space (but elementary row operations may);
(b) elementary row operations do not change linear independence of any set of columns (but elementary column operations may).

Problem 16. The rows of a matrix are linearly independent if and only if the columns span F^k. Similarly the columns are linearly independent if and only if the rows span F^n.

Problem 17. Let A and B be matrices. Show that

$$\mathrm{rk}(A+B) \leq \mathrm{rk}(A) + \mathrm{rk}(B).$$

7. Matrix multiplication

Problem 18. Find two 2×2 matrices A, B such that $AB \neq BA$.

Problem 19. Find a 2×2 matrix A such that $A \neq 0$ but $A^2 = 0$.

Problem 20. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

What is A^k?

8. Roots of unity

Problem 21. Prove that the sum $z_0 + z_1 + \cdots + z_{n-1}$ of all the nth roots of unity is 0 if $n \geq 2$, and 1 if $n = 1$.

Problem 22. For what k is the sum

$$\sum_{i=0}^{n-1} z_i^k = 0?$$

9. Matrices and Linear Maps

Problem 23. Show that if $\phi : V \to W$ is an isomorphism, then for any list of vectors $v_1, \ldots, v_k \in V$ $\mathrm{rk}(v_1, \ldots, v_k) = \mathrm{rk}(\phi(v_1), \ldots, \phi(v_k))$.

Problem 24. Let V, W and Z be vector spaces with bases $\bar{e} = (e_1, \ldots, e_n)$, $\bar{f} = (f_1, \ldots, f_m)$ and $\bar{z} = (z_1, \ldots, z_r)$ respectively. Let $\phi_1 : V \to W$ and $\phi_2 : W \to Z$ be linear maps. Then show that

$$[\phi_2 \circ \phi_1]_{\bar{e},\bar{z}} \;=\; [\phi_2]_{\bar{f},\bar{z}} \cdot [\phi_1]_{\bar{e},\bar{f}}.$$

10. Permutations

Problem 25. Show that, for permutations π and σ,

$$\operatorname{sgn}(\pi\sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma).$$

Problem 26. Show that for any transposition τ, $\operatorname{sgn}(\tau) = -1$.

Problem 27. Show that any permutation $\sigma \in S_n$ can be written as a composition of transpositions.

Problem 28. Show that any permunation $\sigma \in S_n$ can be written as a composition of "neighbor swaps", where a neighbor swap is a transposition of elements i and $i+1$ for some $i \in \{1, \dots, n-1\}$.

11. Determinants

Problem 29. $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$

Problem 30. Prove that if two columns of A are equal, then $\det A = 0$.

Problem 31. Let A^π denote the matrix obtained by permuting the columns of the matrix A according to the permutation π. Then show that $\det(A^\pi) = \operatorname{sgn}(\pi) \cdot \det(A)$.

Problem 32. Suppose A is an $m \times n$ matrix which has an $r \times r$ submatrix with nonzero determinant. Then $\operatorname{rk}(A) \geq r$. In fact, $\operatorname{rk}(A)$ is the maximum of such r.

Problem 33. Let $A \in F^{m\times n}$. Show that A has a right inverse if and only if A has full row-rank, i.e., $\operatorname{rk}(A) = m$. Similarly, show that A has a left inverse if and only if $\operatorname{rk}(A) = n$.

Problem 34. If $m \neq n$, $|F| = \infty$, and A has a right inverse, then A has infinitely many right inverses.

Problem 35. Give a simple explicit formula for

$$\det \begin{pmatrix} a & b & b & \dots & b & b & b \\ b & a & b & \dots & b & b & b \\ b & b & a & \dots & b & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b & b & b & \dots & a & b & b \\ b & b & b & \dots & b & a & b \\ b & b & b & \dots & b & b & a \end{pmatrix}.$$

The resulting expression should be completely factored.

Problem 36. Let $x_1, \dots, x_n \in F$, and define the Vandermonde matrix

$$V(x_1, \dots, x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}.$$

Show that

$$\det V(x_1, \dots, x_n) = \prod_{i<j} (x_j - x_i)$$

Problem 37. What is

$$\det \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix}?$$

Problem 38. Prove the cofactor expansion formula.

Problem 39. If $A, B \in M_n(F)$, then $\det(AB) = \det(A) \cdot \det(B)$.

Problem 40. Prove that for $A \in M_n(F)$, $A \cdot \mathrm{adj}(A) \ = \ \det(A) \cdot I_n$.

12. Eigenvalues, Eigenvectors, the Characteristic Polynomial

Problem 41. For all $A \in M_n(\mathbb{C})$ and for all eigenvalues $\lambda \in \mathbb{C}$, we have

$$\text{algebraic multiplicity of } \lambda \geq \text{geometric multiplicity of } \lambda.$$

Problem 42. Let $A, B \in M_n(F)$ where F is an arbitrary field. Prove: if $A \sim B$, then $f_A(t) = f_B(t)$. (Recall: $A \sim B$ (A, B are similar) means that $(\exists S, S^{-1} \in M_n(F))(B = S^{-1}AS)$.)

Problem 43. The converse to Problem 42 is false. *Hint:* Find A, B such that $\mathrm{rk}(A) \neq \mathrm{rk}(B)$ but $f_A(t) = f_B(t)$.

Problem 44. (a) Consider the matrices

$$A = \begin{pmatrix} 2 & 7 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Are they similar?

(b) Same question for the matrices

$$A = \begin{pmatrix} 2 & 7 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Problem 45. A matrix A is diagonalizable if there exists a matrix S such that $S^{-1}AS = D$, where D is a diagonal matrix. Prove that over $\mathbb{C}$, a matrix A is diagonalizable if and only if for all eigenvalues λ of A, we have

$$\text{alg.mult.}(\lambda) = \text{geom.mult.}(\lambda).$$

Problem 46. Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Problem 47. If $v_1, ..., v_k$ are eigenvectors of A to **distinct** eigenvalues ($v_i \neq 0, Av_i = \lambda_i v_i$, $\lambda_i \neq \lambda_j$ for $i \neq j$), then the v_i are linearly independent.

Problem 48. For $A \in M(\mathbb{C})$, how are the eigenvalues of A and A^T related (both in algebraic and geometric multiplicity)?

Problem 49. Let $f_A(t)$ denote the characteristic polynomial of the matrix A. The Cayley-Hamilton theorem states that $f_A(A) = 0$. Prove this theorem for diagonalizable matrices.

Problem 50. If λ is an eigenvalue of A and $f \in F[x]$, then $f(\lambda)$ is an eigenvalue of $f(A)$. (Is the converse true?)

Problem 51. For $a \in M_n(\mathbb{R})$, (a) define e^A, and (b) prove that if λ is an eigenvalue of A, then e^λ is an eigenvalue of e^A. (Is the converse true?)

Problem 52. Let R_θ denote the matrix of the rotation of the plane by θ. Find an eigenbasis of R_θ over $\mathbb{C}$, and observe that it is independent of θ.

13. Inner Products, Unitary Matrices, Gram-Schmidt Orthogonalization

Problem 53. Prove that $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots \in C[0, 2\pi]$ is an orthogonal system and thus linearly independent under the inner product $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$.

Problem 54. State and prove the Gram-Schmidt orthogonalization theorem in complex Hermitian space case.

Problem 55. A matrix $U \in M_n(\mathbb{C})$ is said to be unitary if its columns form an orthonormal basis for $\mathbb{C}^n$. Prove that a matrix U is unitary if and only if its *rows* form an orthonormal basis for $\mathbb{C}^n$.

Problem 56. If A is unitary and λ is an eigenvalue of A, prove that $|\lambda| = 1$.

14. Normal Matrices

Problem 57. If a triangular matrix is normal, prove it is diagonal.

Problem 58. If A is normal, prove

(1) A is Hermitian iff all eigenvalues of A are real.
(2) A is unitary iff all eigenvalues of A have unit absolute value.

Problem 59. Let $A \in M_n(\mathbb{R})$. Prove that A is similar to a triangular matrix iff A is orthogonally similar to a triangular matrix iff all (complex) eigenvalues of A are real.

15. Spectral Theorem

Problem 60 (Schur's Theorem)**.** For any matrix $A \in M_n(\mathbb{C})$ with eigenvalues $\lambda_1, \dots, \lambda_n$, there exists a unitary matrix U and an upper triangular matrix T such that

$$A \;=\; U^*TU \;,$$

and the diagonal entries of T are $\lambda_1, \dots, \lambda_n$. (Hint: Gram-Schmidt)

Problem 61 (Real Spectral Theorem)**.** Let $A \in M_n(\mathbb{R})$. Prove that A is orthogonally similar to diagonal matrix iff A is symmetric ($A = A^t$).

16. Rayleigh quotients

Problem 62. Let A be an $n \times n$ real symmetric matrix. The *Rayleigh quotient* of A is the $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ function $R_A(x) = x^TAx/x^Tx$. Let the eigenvalues of A be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Prove $\lambda_1 = \max_{x \in \mathbb{R}} R_A(x)$ and $\lambda_n = \min_{x \in \mathbb{R}} R_A(x)$.

Problem 63 (Courant-Fischer Theorem)**.** Let A be an $n \times n$ real symmetric matrix. Let the eigenvalues of A be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$\lambda_k \;=\; \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{x \in S} \frac{x^TAx}{x^Tx} \;=\; \min_{\substack{T \subseteq \mathbb{R}^n \\ \dim(S)=n-k+1}} \max_{x \in T} \frac{x^TAx}{x^Tx}$$

Problem 64 (Eigenvalue Interlacing)**.** Let A be an $n \times n$ real symmetric matrix with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and let B be a $(n-1) \times (n-1)$ principal submatrix of A with eigenvalues $\nu_1 \geq \cdots \geq \nu_{n-1}$. Then show that

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \cdots \geq \mu_{n-1} \geq \nu_{n-1} \geq \mu_n \;.$$

Problem 65 (Sylvester's Law of Intertia)**.** Let $A \in M_n(\mathbb{R})$ be a real-symmetric matrix and let $B \in M_n(\mathbb{R})$ be a non-singular matrix. Then show that the matrices A and BAB^T have the same number of positive, negative and zero eigenvalues.

17. Graphs, Adjacency Matrices and Eigenvalues

Problem 66. Suppose A has eigenvalues $\lambda_1, \dots, \lambda_n$. Prove that $aA + bI$ has eigenvalues $a\lambda_i + b$ with corresponding multiplicities.

Problem 67. Compute all the eigenvalues and eigenvectors for the complete graph K_n on n vertices.

Problem 68. Let C_n denote the cycle graph on n vertices. Show that if ω is an n^{th} root of unity, then the vector $x = (1, \omega, \dots, \omega^{n-1})^T$ is an eigenvector of A_{C_n}. Use this to compute all the (real) eigenvalues and (real) eigenvectors of A_{C_n}.

Problem 69. Assume A is a nonnegative matrix with a positive eigenvector x (all coordinates of x are positive) with eigenvalue λ, i. e., $x \neq 0$ and $Ax = \lambda x$.

(1) Prove that there is a non-singular diagonal matrix S such that

$$S^{-1}AS\mathbf{1} = \lambda\mathbf{1} \;,$$

where $\mathbf{1}$ denotes the vector $(1, \dots, 1)^T$.

(2) Prove that for all eigenvalues μ of A, $|\mu| \leq \lambda$.

(3) Show that the eigenvalue λ has multiplicity 1.
(Note: Part 3 of the above problem is not true as stated (find a counter-example!). It will be true under an additional condition on the matrix - see problem 75)

Problem 70. Let A be the adjacency matrix of a connected undirected graph G. Show that A must have a positive eigenvector. (Hint: Rayleigh quotient)

Problem 71. Prove that $\frac{1}{n}\sum_{i=1}^{n} d(i) \leq \lambda_1 \leq \max_i d(i)$, where $d(i)$ denotes the degree of vertex i.

Problem 72. Let G be a graph such that A_G has eigenvalues $\mu_1 \geq \cdots \geq \mu_n$. Then show that G can be colored with $\lfloor \mu_1 \rfloor + 1$ colors.

Problem 73. Let G be a d-regular graph such that A_G has eigenvalues $d = \mu_1 \geq \cdots \geq \mu_n$. An *independent set* in a graph G is defined as a set $S \subset V$ such that for every edge $(i, j) \in E$, either $i \notin S$ or $j \notin S$.

(1) Show that if $\chi_S \in \mathbb{R}^n$ is the *indicator vector* of an independent set S, such that $\chi_S(i) = 1$ if $i \in S$ and $\chi_S(i) = 0$ otherwise, then $\chi_S^T A_G \chi_S = 0$.
(2) Consider the matrix $B = A_G - cJ$. For what values of c is the smallest eigenvalue of B equal to μ_n?
(3) Use the Rayleigh quotient of χ_S to show that for any independent set S in G,

$$|S| \;\leq\; \left(\frac{\mu_n}{d - \mu_n}\right) \cdot n \; .$$

(4) Use this to show that coloring G requires at least $1 + \frac{d}{-\mu_n}$ colors.

Problem 74. Suppose an undirected graph has sorted eigenvalues $\mu_1 \geq \cdots \geq \mu_n$. Prove

(1) $(\forall i)(|\mu_i| \leq \mu_1)$
(2) If the graph G is connected, then $(\forall i \geq 2)(\mu_i < \mu_1)$
(3) If the graph G is connected, then $|\mu_n| = \mu_1$ iff G is bipartite.
(4) If G is a bipartite graph, then $(\forall i)(\mu_i = -\mu_{n-i+1})$.

A *directed graph* G is called **strongly connected** if for every pair $(i.j)$ of vertices, i is reachable from j via a directed path and vice-versa.

With a matrix $A \in \mathbb{R}^{n\times n}$, we associate a directed graph G_A on n vertices by including the (directed) edge (i, j) if and only if $A_{ij} \neq 0$. The matrix A is called **irreducible** if G_A is strongly connected.

Problem 75. (Problem 69 continued) Assume A is a nonnegative irreducible matrix with a positive eigenvector x (all coordinates of x are positive) with eigenvalue λ, i. e., $x \neq 0$ and $Ax = \lambda x$. Show that the eigenvalue λ has multiplicity 1.

18. Random Walks

The following problems extend the analysis done in class for regular graphs, to arbitrary undirected (and connected) graphs.

Problem 76. Let G be an undirected graph with adjacendy matrix A_G and let $M = A_G D^{-1}$ be the diffusion matrix for the simple random walk on G (here D^{-1} is a diagonal matrix with $D^{-1}_{ii} = 1/\deg(i)$). If λ is any eigenvalue of M, then show that $|\lambda| \leq 1$.

Problem 77. For an undirected connected graph G, show that the diffusion matrix $M = AD^{-1}$ is similar to the matrix $M' = D^{-1/2}AD^{-1/2}$. Use this to show that all eigenvalues of M are real and it is possible to find real eigenvectors for each eigenvalue.

Problem 78. Let G be a connected undirected graph. Show that the simple random walk on G has a unique stationary distribution. (Hint: Use problems 69, 70 and 75)

Problem 79. For an undirected and connected graph G, show that the distribution $\pi(i) = \deg(i)/(\sum_{j=1}^{n} \deg(j))$ is a stationary distribution for the simple random walk on G.

Problem 80. Let G be an undirected and connected graph and let the distribution π be as defined above. Let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the diffusion matrix AD^{-1} and let $\lambda = \max\{\lambda_2, -\lambda_n\}$. Show that if we start a simple random walk in G at vertex i_0, then after t steps we have that for each $i \in V$,

$$\left|p^{(t)}(i) - \pi(i)\right| \;\leq\; \sqrt{\frac{\deg(i)}{\deg(i_0)}} \cdot \lambda^t \;.$$

19. Laplacian Matrices

The problems below deal only with undirected graphs. The (combinatorial) Laplacian is defined as the matrix $L = D - A$, where D is a diagonal matrix with entries $D_{ii} = \deg(i)$ and A is the adjacendy matrix. The normalized Laplacian is defined as the matrix $N = I - D^{-1/2}AD^{-1/2}$.

Problem 81. For an undirected graph G, show that

(1) the multiplicity of the eigenvalue 0 for the Laplacian matrix (L) equals the number of connected components of G.
(2) the multiplicity of the eigenvalue 0 for the normalized Laplacian matrix (N) equals the number of connected components of G.

20. Positive (semi)definite matrices

Problem 82. A real symmetric matrix B is said to be *positive semidefinite* if $x^T Bx \geq 0$ for all $x \in \mathbb{R}^n$. Show that the following are equivalent for a symmetric matrix $B \in \mathbb{R}^{n\times n}$.

(1) B is positive semidefinite.
(2) All eigenvalues of B are non-negative.
(3) For some $m \in \mathbb{N}$, there exists a matrix $V \in \mathbb{R}^{m\times n}$ such that $B = V^T V$.
(4) For some $m \in \mathbb{N}$, there exist vectors $v_1, \ldots, v_n \in \mathbb{R}^m$ such that for each $i, j \in [n]$, $B_{ij} = \langle v_i, v_j\rangle$.

Problem 83. Show that for any graph G, both the Laplacian matrix L and the normalized Laplacian matrix N are positive semidefinite.

Problem 84. A real symmetric matrix B is said to be *positive definite* if $x^T Bx > 0$ for all $x \in \mathbb{R}^n$. Show that the following are equivalent for a symmetric matrix $B \in \mathbb{R}^{n\times n}$.

(1) B is positive definite.
(2) All eigenvalues of B are positive.
(3) All corner determinants of B are positive, i. e., $(\forall k \leq n)(\det((B_{ij})_{i,j\leq k}) > 0$.

Problem 85. Let $B \in \mathbb{R}^{n\times n}$ be a symmetric positive semidefinite matrix. Let $x_1, \ldots, x_k \in \mathbb{R}^n$ be orthogonal vectors and $\alpha \geq 0$ be such that for each $i \in [k]$, $R_B(x_i) \leq \alpha$. Then show that for any $x \in Span(x_1, \ldots, x_k)$, $R_B(x) \leq k \cdot \alpha$.

21. Miscellaneous Problems

Problem 86. Let R_1 and R_2 be two random variables (depending on the same underlying space of outcomes to a random experiment) and let $\alpha \in \mathbb{R}$ be such that such that $\frac{\mathbb{E}[R_1]}{\mathbb{E}[R_2]} \leq \alpha$. Also, assume that $R_2 > 0$ for *every outcome* of the random experiment. Then show that there exists an outcome of the random experiment for which $\frac{R_1}{R_2} \leq \alpha$.
(Note: In class this problem was stated with only the assumption that $\mathbb{E}[R_2] > 0$, which is insufficient to obtain the required conclusion. However, we can obtain the desired conclusion if we assume that $R_2 > 0$ for every outcome of the random experiment.)