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1 More on Adjacency Matrices

Recall that we have $G = (V, E)$ and its adjacency matrix A and eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ and $\sum \mu_i = tr(A) = 0$.

Exercise 1.1

1. *Show that the following holds*

$$\frac{1}{n}\sum_{i \in V} deg(i) \leq \mu_1 \leq \max_{i \in V} deg(i)$$

2. *If G has a positive eigenvector with eigenvalue λ, then for all other eigenvalues μ, with $|\mu| \leq \lambda$.*

3. *If G is connected, then A_G has a positive eigenvector with eigenvalue μ_1. (Use Rayleigh quotient)*

Exercise 1.2 *If G has A_G with eigenvalues, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, then G can be colored with $\lfloor \mu_1 \rfloor + 1$ colors.*

This exercise was discussed in class. We follow the same proof scheme as before, for showing that a graph with maximum degree d can be colored with $d+1$ colors. We proceed by induction on the number of vertices in G. The case with $n = 1$ is trivial since the only eigenvalue is 0 and the graph can be colored with 1 color.

For the case with n vertices, we know (from the previous exercise) that $\mu_1 \geq \frac{1}{n}\sum_{i \in V} deg(i)$. Thus, there must be a vertex i with degree at most μ_1. Since degrees are integers, we must have $deg(i) \leq \lfloor \mu_1 \rfloor$. Consider the graph G' (on $n-1$ vertices) obtained by removing the vertex i from G. Use Rayleigh quotients to prove that if ν_1 is the largest eigenvalue of G', then $\nu_1 \leq \mu_1$. By induction, G' can be colored with $\lfloor \nu_1 \rfloor + 1 \leq \lfloor \mu_1 \rfloor + 1$ colors. Since the vertex i we removed has at most $\lfloor \mu_1 \rfloor$ neighbors, we can assign it a color which is different from the colors of all its neighbors. This gives a valid coloring of G.

Exercise 1.3 *Suppose G' is generated from G via removing a vertex. Let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of G and let $\nu_1 \geq \cdots \geq \nu_{n-1}$ be the eigenvalues of G'. Then use Rayleigh quotients to show that*

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \ldots \geq \nu_{n-1} \geq \mu_n\,.$$

2 Random Walk on Graphs

Given a starting vertex $i_0 \in V$, a simple random walk on the graph $G = (V, E)$ is the following process:

- Start at the given vertex i_0.
- At each step, pick a random neighbor of the current vertex and move to neighbor vertex.

Consider a vector $p^{(t)}$ where $p^{(t)}(j)$ is supposed to denote the chance that the random walk after t steps, is at vertex j. If the starting vertex is i, then $p^{(0)} \in \mathbb{R}^n$ is a vector with $p^{(0)}(j) = 1$ if $j = i_0$ and 0 otherwise. In general a probability distribution p over vertices must have the property that $\sum_j p(j) = 1$ and $p(j) \geq 0 \ \forall j \in V$.

To understand the distribution $p^{(t+1)}$ in terms of $p^{(t)}$, we note that if at step $t+1$ we are to land at a vertex i, then at step t we must be at some j which is a neighbor of j (which we denote by $j \sim i$). We can then write

$$p^{(t+1)}(i) \ = \ \sum_{j \sim i} p^{(t)}(j) \cdot \frac{1}{deg(j)} \, .$$

In matrix form, this gives

$$p^{(t+1)} = M \cdot p^{(t)} \, ,$$

where M is a matrix with entries

$$M_{ij} = \begin{cases} \frac{1}{deg(j)} & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases} \, .$$

Note that $M = A_G D^{-1}$ where D is a diagonal matrix with entries $D_{ii} = deg(i)$ and A is the adjacency matrix. The matrix M is also referred to as the *diffusion* matrix for the simple random walk on G. We can use linear algebra to analyze random walks once we notice that the distribution after t steps can be written as

$$p^{(t)} \ = \ M^t \cdot p^{(0)} \, .$$

In particular, we will be interested in the question of how fast the random walk reaches a *stationary distribution* i.e., a distribution which does not change as the random walk proceeds.

Definition 2.1 (Stationary distribution) *A distribution π is a stationary distribution for a random walk with diffusion matrix M if*

$$M\pi = \pi \, .$$

Thus, π is simply a non-negative eigenvector of M with eigenvalue 1, which is multiplied by an appropriate positive constant to ensure that $\sum_i \pi(i) = 1$. For all graphs, the random walk on G will have a stationary distribution (we will prove this in the problem set), but not all walks might reach the stationary distribution if started from an arbitrary vertex. For example, if G has many connected components, then a random walk will stay in it's own connected component. Also, G is bipartite, then a walk will oscillate between the two sides. We will show in the analysis below that these are essentially the only two obtacles to reaching a stationary distribtion.

2.1 Random walks on regular graphs

One problem in applying some of the theory from previous lectures is that the matrix M is not symnmetric. However, if G is d-regular (each vertex has degeee d), then the matrix M becomes

$$M = AD^{-1} = \frac{1}{d} \cdot A\,,$$

and is a symmetric matrix. We will see later how to extend the ideas to general graphs. We will also assume that the graph G is connected, as otherwise we can analyze the walk in each connected component separately. Then the eigenvalues for the matrix M are $\mu_1/d, \ldots, \mu_n/d$ and the eigenvectors are the same as those for the matrix A.

Exercise 2.2 *Suppose G is d-regular. Then $x = (1/n, ..., 1/n)^T$ is a stationary distribution for the simple random walk on G.*

Let $\mu = \max_{i=2,\ldots,n} |\mu_i| = \max\{\mu_2, -\mu_n\}$. We will show that the distribution of the random walk converges to the stationary distribution as long as $\mu < d$. Recall that if G is connected iff $\mu_2 < d$ and non-bipartite iff $\mu_n > -d$.

Lemma 2.3 *Let G be a d-regular graph and let $\mu = \max\{\mu_2, -\mu_n\}$. Then, after t steps of a simple random walk on G started at an arbitrary vertex i_0, we have that*

$$\forall i \in V, \quad \left|p^{(t)}(i) - 1/n\right| \leq \left(\frac{\mu}{d}\right)^t .$$

Proof: Let $M = \frac{1}{d}A$ such that $p^{(t)} = M^t p^{(0)}$. Let $u_1, ..., u_n$ be the orthonormal eigenbasis of M such that we can write $p^{(0)} = \sum_{i=1}^n \alpha_i u_i$.

$$\begin{aligned} p^{(t)} &= M^t p^{(0)} \\ &= M^t (\sum_i \alpha_i u_i) \\ &= \sum_i \alpha_i (\frac{\mu_i}{d})^t u_i \quad (\textit{Since } M^t u_i = (\frac{\mu_i}{d})^t u_i) \end{aligned}$$

What is u_1? Setting $u_1 = c(1/n, ..., 1/n)^T$, $\langle u_1, u_1 \rangle = \sum_i c^2 (1/n)^2 = 1$ gives us $u_1 = (1/\sqrt{n}, ..., 1/\sqrt{n})^T$. What is α_1? Since $p^{(0)} = \sum_i \alpha_i u_i$ and $u_1, ..., u_n$ form an orthonormal basis,

$$\alpha_1 = \left\langle p^{(0)}, u_1 \right\rangle = \frac{1}{\sqrt{n}} \sum_i p_i^{(0)} = \frac{1}{\sqrt{n}}$$

So we have

$$\begin{aligned} p^{(t)} &= M^t p^{(0)} \\ &= \frac{1}{\sqrt{n}} \left(\frac{d}{d}\right)^t \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{\sqrt{n}} \end{pmatrix} + \alpha_2 \left(\frac{\mu_2}{d}\right)^t u_2 + ... + \alpha_n \left(\frac{\mu_n}{d}\right)^t u_n \end{aligned}$$

Thus we have

$$p^{(t)} - \begin{pmatrix} \frac{1}{n} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{n} \end{pmatrix} = \alpha_2 \left(\frac{\mu_2}{d}\right)^t u_2 + ... + \alpha_n \left(\frac{\mu_n}{d}\right)^t u_n$$

Let e denote the "error vector" $\alpha_2 \left(\frac{\mu_2}{d}\right)^t u_2 + ... + \alpha_n \left(\frac{\mu_n}{d}\right)^t u_n$. We need to show that for each $i \in V$, $|e(i)| \leq (\mu/d)^t$. The following claim finishes the proof.

Claim 2.4 $\|e\| = \sqrt{\sum_i |e(i)|^2} \leq \frac{\mu}{d}$.

Proof:

$$\|e\|^2 = \langle e, e \rangle = \sum_{i=2}^{n} \alpha_i^2 \left(\frac{\mu_i}{d}\right)^2 \leq \left(\sum_{i=2}^{n} \alpha_i^2\right) \cdot \left(\frac{\mu}{d}\right)^2 \leq \left(\frac{\mu}{d}\right)^2 .$$

Here the last inequality follows from the fact that $\left\langle p^{(0), p^{(0)}} \right\rangle = \sum_{i=1}^{n} \alpha_i^2 = 1$. ■

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2.2 Random walks on general undirected graphs

For general graphs G, we have $M = AD^{-1}$ as defined above. Let $D^{-\frac{1}{2}}$ be the diagonal matrix with entries $(D^{-\frac{1}{2}})_{ii} = \frac{1}{\sqrt{deg(i)}}$. The analysis for random walks is very similar to the above, but we use the matrix $D^{-1/2}AD^{-1/2}$ which is similar to the matrix $M = AD^{-1}$.

Exercise 2.5 *Show that* $AD^{-1} = M \sim D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$.

Thus, the eigenvalues of the two matrices are the same there is an isomorphic between their eigenspaces for each eigenvalue. However, the matrix $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ is symmetric and has real eigenvalues and an orthonormal basis of real eigenvectors.

Random walks on a connected undirected graph can be analyzed by expressing the initial distribution in terms of the eigenvectors of the matrix $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. We will leave the details to the problem set.