REU 2013: Apprentice Program **Summer 2013**

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1 Eigenvalues and eigenvectors of adjacency matrices

Recall given an adjacency matrix for an undirected graph $G = (V, E)$, A is real and symmetric, and hence diagonalizable ($A = UDU^*$, D is diagonalizable with real entries). We can decompose A as follows:

$$A = \sum_{i=1}^{n} \mu_i u_i u_i^T$$

where we order the eigenvalues so that $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$.

Exercise 1.1 *Given an adjacency matrix A, $(A^l)_{ij} = \#$ of paths of length l between i and j.*

Definition 1.2 (Trace) *Given an $n \times n$ matrix A, $Tr(A)$ is the sum of its diagonal entries.*

Exercise 1.3 *Suppose G is a graph with n vertices, m edges, and no self-loops. Then show that*

1. $\sum_{i=1}^{n} \mu_i = 0$ *(Hint: sum of the roots of the characteristic polynomial).*
2. $\sum_{i=1}^{n} \mu_i^2 = 2m$ *(Hint: what are the eigenvalues of A^2?)*

Exercise 1.4 *If $B \sim A$, then $Tr(B) = Tr(A)$.*

Definition 1.5 (Clique) *$K_n \equiv$complete graph on n vertices.*

$$A_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

Exercise 1.6 *Find the eigenvalues and eigenvectors of the complete graph on n vertices, denoted as K_n. In K_n, we have an edge between any two vertices i and j such that $i \neq j$.*
Hint: $A_{K_n} = J - I$ where J denotes the $n \times n$ matrix with all entries equal to 1. Look at the dimension of the null space of J.

Exercise 1.7 *Let $C_n \equiv$ cycle of length n. What is its adjacency matrix? Find the eigenvalues and eigenvectors.*

Hint: Consider $v = (1, \omega, \omega^2, ..., \omega^{n-1})^T$, where ω is a root of the unity.

Definition 1.8 *A graph G is said to be d-regular if $\forall i \in V, deg(i) = d$.*

Note that for any d-regular graph G, the vector $(1, \ldots, 1)^T$ is an eigenvector of A_G with eigenvalue d. The graphs K_n and C_n are respectively $(n-1)$-regular and 2-regular.

2 Rayleigh Quotients

Definition 2.1 (Rayleigh Quotient) *For a real matrix A, the Rayleigh quotient of A, denoted as $R_A(\cdot)$ is a function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$, defined as follows*

$$R_A(x) = \frac{x^T A x}{x^T x}\,.$$

We know that $A = \sum_{i=1}^n \mu_i U_i U_i^T$ where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The eigenvalues of a real-symmetric matrix A can also be characterized in terms of the Rayleigh quotients. For example, we can show that

$$\mu_1 = \max_{x \in \mathbb{R}^n \setminus \{0\}} R_A(x)$$
$$\mu_n = \min_{x \in \mathbb{R}^n \setminus \{0\}} R_A(x)$$

First note that if v_i is the eigenvector corresponding to the eigenvalue μ_i, then $R_A(v_i) = \mu_i$. Thus, we have that

$$\max_{x \in \mathbb{R}^n} R_A(x) \;\geq\; R_A(v_1) \;=\; \mu_1\,.$$

Also, for any $x \in \mathbb{R}^n \setminus \{0\}$, we can express it in the orthonormal basis given by the eigenvectors $v_1, \ldots, v_n$ of A. Writing $x = \sum_i \alpha_i v_i$ gives

$$x^T x \;=\; \left(\sum_{i=1}^n \alpha_i v_i\right)^T \left(\sum_{i=1}^n \alpha_i v_i\right) \;=\; \sum_{i,j} \alpha_i \alpha_j v_i^T v_j \;=\; \sum_{i=1}^n \alpha_i^2\,.$$

Similarly, we get

$$x^T A x \;=\; \left(\sum_{i=1}^n \alpha_i v_i\right)^T \left(\sum_{i=1}^n \alpha_i A v_i\right) \;=\; \left(\sum_{i=1}^n \alpha_i v_i\right)^T \left(\sum_{i=1}^n \alpha_i \mu_i v_i\right) \;=\; \sum_{i=1}^n \alpha_i^2 \cdot \mu_i\,.$$

Now we can express $R_A(x)$ as

$$R_A(x) \;=\; \frac{x^T A x}{x^T x} \;=\; \frac{\sum_{i=1}^n \alpha_i^2 \cdot \mu_i}{\sum_{i=1}^n \alpha_i^2 \mu_i}$$

Using the fact that $\mu_i \leq \mu_1$ for all i, we get that $R_A(x) \leq \mu_1$ for all x. Combining it with $R_A(v_1) = \mu_1$, we get that $\mu_1 = \max_{x \in \mathbb{R}^n \setminus \{0\}} R_A(x)$. Similarly, we can prove that $\mu_n = \min_{x \in \mathbb{R}^n \setminus \{0\}} R_A(x)$
Similarly, we can observe that

$$\mu_2 = \max_{x \in Span(v_2, \ldots, v_n) \setminus \{0\}} \frac{x^T A x}{x^T x}$$

Note that saying that $x \in Span(v_2, \ldots, v_n)$ is equivalent to saying that $\langle x, v_1 \rangle = x^T v_1 = 0$.

Exercise 2.2 *If $v_1, \ldots, v_n$ are orthogonal vectors in $\mathbb{C}^n$ (or in $\mathbb{R}^n$), then for any $k \le n$*

$$\{x \mid \langle x, v_1 \rangle = \cdots = \langle x, v_k \rangle = 0\} \;=\; Span(v_{k+1}, \ldots, v_n)\,.$$

The Courant-Fischer theorem charactertizes all the eigenvalues in terms of the Rayleigh quotients.

Theorem 2.3 (Courant-Fischer) *Let A be an $n \times n$ real symmetric matrix. Let the eigenvalues of A be $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Then*

$$\lambda_k \;=\; \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{x \in S \setminus \{0\}} \frac{x^T A x}{x^T x} \;=\; \min_{\substack{T \subseteq \mathbb{R}^n \\ \dim(S)=n-k+1}} \max_{x \in T \setminus \{0\}} \frac{x^T A x}{x^T x}$$

Try proving the theorem by extending the proof for the characterization of μ_1.

Exercise 2.4 *For a graph G, prove the following*

$$\frac{1}{n} \sum_{i=1}^{n} deg(i) \le \mu_1 \le \max_{i \in V} deg(i)$$

Exercise 2.5 *Show that if G is an undirected graph with eigenvalues $\mu_1 \ge \cdots \ge \mu_n$, then G can be colored with $\lfloor \mu_1 \rfloor + 1$ colors.*

Exercise 2.6 (Interlacing theorem) *Use the Courant-Fischer to show the following: if G has eigenvals $\mu_1 \ge \ldots \ge \mu_n$ and G' is a graph obtained by removing a vertex from G with eigenvalues $\nu_1 \ge \ldots \ge \nu_{n-1}$ then*

$$\mu_1 \ge \nu_1 \ge \mu_2 \ge \ldots \ge \nu_{n-1} \ge \mu_n\,.$$