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Madhur Tulsiani Scribe: Young Kun Ko, David Kim

1 Gram-Schmidt Orthononalization

Exercise 1.1 *If U is unitary and λ is a (complex) eigenvalue, prove that $|\lambda| = 1$.*

Recall that $b_1, \ldots, b_k$ form an orthonormal basis if

$$\langle b_i, b_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Also it is easy to check that $b_1, \ldots, b_k$ are such that $\langle b_i, b_j \rangle = 0$ for $i \neq j$ then $b_1, \ldots b_k$ are linearly independent.

Theorem 1.2 (Gram-Schmidt Orthogonalization) *If $v_1, \ldots, v_k$ are linearly independent, then there exist vectors $b_1, \ldots, b_k$ such that*

1. *$b_1, \ldots, b_k$ form an orthonormal basis.*
2. *For all $i \leq k$, $Span(b_1, \ldots, b_i) = Span(v_1, \ldots, v_k)$.*

Proof: The proof is algorithmic and we claim the following process constructs $b_1, \ldots, b_k$ with the above property:

- Start with $b_1 = \frac{v_1}{\|v_1\|}$.
- For each $i = 2, \ldots, k$, set

$$v_i' \;=\; v_i - \langle v_i, b_1 \rangle \cdot b_1 - \cdots - \langle v_i, b_{i-1} \rangle \cdot b_{i-1} \quad \text{and} \quad b_i' \;=\; \frac{v_i'}{\|v_i'\|}$$

Prove by induction on i that the vectors $b_1, \ldots, b_k$ satisfy the above properties. ∎

2 The Spectral Theorem

We can use Gram-Schmidt orthogonalization to prove the following theorem for *every* square matrix A. The spectral theorem then follows as an easy corollary. Recall that the chracteristic polynomial of any matrix $A \in M_n(\mathbb{C})$ has n complex roots.

Theorem 2.1 (Schur's Theorem) *Let* $\forall A \in \mathbb{C}^{n\times n}$ *be a matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Then there exists a unitary matrix* U *and an upper-triangular matrix* T *such that* $A = UTU^*$. *Also, the diagonal entries of* T *are equal to* $\lambda_1, \ldots, \lambda_n$.

Proof Sketch: A must have at least one eigenvector corresponding to the eigenvalue λ_1. Consider u_1 such that $Au_1 = \lambda_1 u_1$ and $\|u_1\| = 1$. Complete it to a basis $u_1, v_2, \ldots, v_n$ for $\mathbb{C}^n$ and use Gram-Schmidt to obtain an orthonormal basis. Note that the first element of basis given by Gram-Schmidt will still be u_1.

Let the orthonormal basis be $u_1, \ldots, u_n$ and consider a unitary matrix U_1 with columns $u_1, \ldots, u_n$. Then $U_1^* A U_1$ must be of the form

$$U_1^* A U_1 = \left(\begin{array}{c|c} \lambda_1 & B_1 \\ \hline 0 & \\ \vdots & A_1 \\ 0 & \end{array}\right)$$

Since $A \sim U_1^* A U_1$, the characteristic polynomial of $U_1^* A U_1$ must be the same as that of A. Also, since $\det(tI - U_1^* A U_1) = (t - \lambda_1) \cdot \det(tI - A_1)$, A_1 must have eigenvalues $\lambda_2, \ldots, \lambda_n$. As above, we can find a unitary matrix $V_2 \in M_{n-1}(\mathbb{C})$ such that

$$V_2^* A_1 V_2 = \left(\begin{array}{c|c} \lambda_2 & B_2 \\ \hline 0 & \\ \vdots & A_2 \\ 0 & \end{array}\right)$$

Take U_2 to be the matrix

$$U_2 = \left(\begin{array}{c|c} 1 & 0 \quad \cdots \quad 0 \\ \hline 0 & \\ \vdots & V_2 \\ 0 & \end{array}\right),$$

and note that $U_2^* U_1^* A U_1 U_2$ is of the form

$$U_2^* U_1^* A U_1 U_2 = \left(\begin{array}{cc|c} \lambda_1 & * & B_3 \\ 0 & \lambda_2 & \\ \hline \vdots & & A_3 \\ 0 & & \end{array}\right),$$

Continuing this process, we get unitary matrices $U_1, \ldots, U_n$ such that

$$U_n^* \cdots U_1^* A U_1 \cdots U_n = T\,,$$

where T is upper triangular with diagonal entries $\lambda_1, \ldots, \lambda_n$. Taking $U = U_1 \cdots U_n$ proves the theorem. □

A matrix A is called *normal* if $AA^* = A^*A$. Schur's theorem gives the spectral theorem for normal matrices as an easy corollary.

Corollary 2.2 (Spectral Theorem) *Let A be a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\exists U$ such that $A = UDU^*$ where D is a diagonal with entries $\lambda_1, \ldots \lambda_n$.*

Proof: Assuming Schur's theorem, proceed in following steps.

1. If $A = UTU^*$ as in Schur's theorem, then $TT^* = T^*T$.
2. Show that if T is a triangular matrix which is normal, then T must be diagonal.

■

Example 2.3 *Suppose A is normal and thus $A = UDU^*$. Let $U = [u_1, ..., u_n], D_{ii} = \lambda_i$. Then $Au_i = \lambda_i u_i$, and the u_i's form an orthornormal basis of eigenvectors.*

Exercise 2.4 *Show that A is diagonalizable iff for all eigenvalues, algebraic multiplicity = geometric multiplicity.*

Exercise 2.5 $A = UDU^* \Rightarrow A = \sum_{i=1}^{n} \lambda_i u_i u_i^*$.

Definition 2.6 (Hermitian and Symmetric Matrices) *An $n \times n$ matrix A is Hermitian if $A^* = A$. A is called symmetric if $A^T = A$.*

Note that a Hermitian matrix is normal. Also, a real and symmetric matrix is Hermitian. Using the Spectral Theorem gives the following important conclusion for Hermitian matrices.

Proposition 2.7 *Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix. Then all eigenvalues of A must be real.*

Proof: If A is Hermitian, then using $A = UDU^*$ and $A = A^*$ gives $D = D^*$. Since D is diagonal with entries $\lambda_1, \ldots, \lambda_n$, we get that for each $i \in [n]$, $\lambda_i = \overline{\lambda_i}$, which means that all eigenvalues must be real. ■

Also, note that for a real symmetric matrix A, all the eigenvalues of A are real and the proof of Schur's theorem can be carried out over $\mathbb{R}^n$. This gives that a real symmetric matrix can be written as

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T ,$$

where $\lambda_1, \ldots, \lambda_n$ are real eigenvalues and $u_1, \ldots, u_n \in \mathbb{R}^n$ form an orthonormal basis.

3 Adjacency matrices of graphs

Definition 3.1 (Adjcency Matrix) *Let $G = (V, E)$ be a graph. Then the adjacency matrix A is defined as*

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

which is symmetric if G is undirected.

If A is the adjacency matrix of an undirected graph G then all eigenvalues of A are real. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues sorted so that $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$. As before, we can find vectors $u_1, \ldots, u_n \in \mathbb{R}^n$ forming an orthonormal basis such that

$$A = \sum_{i=1}^{n} \mu_i u_i u_i^T .$$

Note that this implies $Au_i = \mu_i u_i$ and thus $u_1, \ldots, u_n$ are orthonormal (real) eigenvectors corresponding to (real) eigenvalues $\mu_1, \ldots, \mu_n$.