**REU 2013: Apprentice Program** | **Summer 2013**

# Lecture 6: July 5, 2013

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## 1 Eigenvalues & Eigenvectors

**Definition 1.1 (Eigenvalue & Eigenvector)** *Let $A \in \mathbb{C}^{n \times n}$. Then $\lambda \in \mathbb{C}$ is said to be an eigenvalue of $A$ if $\exists v \neq 0$ such that*

$$Av = \lambda v \iff (A - \lambda I)v = 0v\,.$$

*Such a $v$ is an called eigenvector of $A$ with eigenvalue $\lambda$.*

**Exercise 1.2** *$\lambda$ is an eigenvalue of $A$ if and only if $det(\lambda I - A) = 0$*

**Exercise 1.3** *Let $U_\lambda = \{v : Av = \lambda v\}$. Prove: $U_\lambda$ is a subspace of $\mathbb{C}^n$.*

**Definition 1.4 (Characteristic Polynomial)** *$f_A(t) = det(tI - A)$ is called the characteristic polynomial of $A$.*

From the above, we know that $\lambda$ is an eigenvalue of $A$ iff *lambda* is a root of the characteristic polynomial $f_A(t)$.

**Example 1.5** *Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The characteristic polynomial is $f_A(t) = \det(tI - A) = (t-1)^2$. The only eigenvalue is $\lambda = 1$, and $U_\lambda = \mathbb{C}^2$, since $Av = Iv = v$ for all $v \in \mathbb{C}^2$.*

**Example 1.6** *Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The characteristic polynomial is still $f_A(t) = \det(tI - A) = (t-1)^2$ and the only eigenvalue is $\lambda = 1$. However, $U_\lambda$ is now only the one-dimensional space*

$$U_\lambda \;=\; \left\{ \alpha \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\mid\; \alpha \in \mathbb{C} \right\}.$$

**Exercise 1.7** *Calculate the eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.*

Note that the eigenvalue of a real matrix may be complex valued. Also, any polynomial $p(t)$ of degree $n$ over $\mathbb{C}$ can be factored as $c(t - \lambda_1)...(t - \lambda_n)$ for $c, \lambda_1, ..., \lambda_n \in \mathbb{C}$. Thus, the characteristic polynomial of a matrix always has $n$ (not necessarily distinct) complex roots.

**Definition 1.8 (Algebraic Multiplicity)** *The algebraic multiplicity of an eigenvalue $\lambda$ is the number of times $t - \lambda$ appears as a factor in the characteristic polynomial $f_A(t)$.*

**Definition 1.9 (Geometric Multiplicity)** *The geometric multiplicity of an eigenvalue $\lambda$ is $dim(U_\lambda)$.*

**Exercise 1.10** *Algebraic multiplicity $\geq$ geometric multiplicity.*

Note that the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives an example where the algebraic multiplicity is strictly greater than the geometric multiplicity of an eigenvalue.

**Exercise 1.11** *Let $Av = \lambda v$ for $v \in \mathbb{C}^n, A \in \mathbb{R}^{n \times n}, \lambda \in \mathbb{R}$. Then $Re(v), Im(v)$ are also eigenvectors with eigenvalue $\lambda$.*

Thus, if a real matrix $A$ has an eigenvector with a real eigenvalue $\lambda \in \mathbb{R}$, then it also has a real eigenvector with the same eigenvalue.

**Example 1.12** *Calculate the eigenvalues, eigenvectors, and the characteristic polynomial for the rotation matrix, $R_\theta = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$.*

$f_{R_\theta}(t) = (t - cos\theta)^2 + (sin\theta)^2 = 0$ gives $t = cos\theta \pm isin\theta = e^{\pm i\theta}$. Let $\lambda_1 = cos\theta + isin\theta$, $\lambda_2 = cos\theta - isin\theta$. Then $U_{\lambda_1} = \{\alpha \begin{pmatrix} i \\ 1 \end{pmatrix} : \alpha \in \mathbb{C}\}$, $U_{\lambda_2} = \{\alpha \begin{pmatrix} 1 \\ i \end{pmatrix} : \alpha \in \mathbb{C}\}$. Note that the eigenvectors do not depend on $\theta$.

**Exercise 1.13** *If $v_1, ..., v_n$ are eigenvectors with distinct eigenvalues $\lambda_1, ..., \lambda_n$, then $v_1, ..., v_n$ are linearly independent.*

**Definition 1.14 (Similar Matrices)** *$A$ and $B$ are similar, $A \sim B$, if $\exists S \in \mathbb{C}^{n \times n}$ such that $A = S^{-1}BS$.*

**Definition 1.15 (Diagonalizable Matrices)** *$A$ is diagonalizable if $A = S^{-1}DS$ where $D$ is a diagonal matrix. (A is similar to a diagonal matrix).*

**Exercise 1.16** *Prove that similarity between matrices is an equivalence relation.*

**Exercise 1.17** *Prove: If $A \sim B$, then $f_A(t) = f_B(t)$. Thus, if $A \sim B$, then they have the same eigenvalues and each eigenvalue has the same algebraic multiplicity for both $A$ and $B$.*

**Exercise 1.18** *Let $A \sim B$. Then for each $\lambda$ which is an eigenvalue of $A$ (and hence also of $B$), show that $U_\lambda^{(A)}$ is isomorphic to $U_\lambda^{(B)}$. Thus, each eigenvalue also has the same geometric multiplicity for both $A$ and $B$.*

This follows by noting that $S^{-1}BSv = \lambda v \Rightarrow BSv = \lambda Sv$. Hence, $v \mapsto Sv$ is a bijective linear map from $U_\lambda^{(A)}$ to $U_\lambda^{(B)}$.

## 2 Inner Products and Unitary Matrices

**Definition 2.1 (Inner Product)** *Let $u, v \in \mathbb{C}^n$. Then the Hermitian inner product of $u$ and $v$ is defined as*

$$\langle u, v \rangle \; = \; \sum_{i=1}^{n} \overline{u_i} v_i \,,$$

*where $\overline{u_i}$ denotes the conjugade of $u_i$. Note that this the same as the usual dot-product if $u, v \in \mathbb{R}^n$.*

Note that the quantity $\langle u, u \rangle = \sum_i |u_i|^2$ is always non-negative and is zero only when $u = 0$. We define $\|u\| = \sqrt{\langle u, u \rangle}$, which extends the notion of *length* of a vector, to vectors in $\mathbb{C}^n$.

**Definition 2.2 (Orthogonal Vectors)** *Two vectors $u, v$ are said to be orthogonal if $\langle u, v \rangle = 0$.*

**Definition 2.3 (Orthonormal Basis)** *$\{v_1, ..., v_k\}$ is an orthonormal basis for $V$ if it is a basis such that*

$$\forall i, j \quad \langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \;.$$

**Example 2.4** *For the rotation matrix $R_\theta$, the eigenvectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ i \end{pmatrix}$ are orthogonal. After scaling, the vectors to $\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$, we obtain an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of $R_\theta$.*

**Definition 2.5 (Unitary Matrix)** *A matrix $U$ is called a unitary matrix if the columns of $U$ form an orthonormal basis of $\mathbb{C}^n$.*

**Definition 2.6 (Adjoint of a matrix)** *For a matrix $A \in M_n(\mathbb{C})$, its adjoint, denoted as $A^*$ is the matrix defined as $(A^*)_{ij} = \overline{a_{ji}}$ i.e., $A^* = \overline{A}^T$.*

Note that it follows from the fact that $(AB)^T = B^T A^T$ that we have $(AB)^* = B^* A^*$.

**Exercise 2.7** *$U$ is unitary if and only if $U^* U = U U^* = I$.*

Note that $(U^* U)_{ij} = \left\langle U^{(i)}, U^{(j)} \right\rangle$, where $U^{(i)}$ denotes the $i$th column of $U$. Hence $U^* U = I$ if and only if the columns form an orthonormal basis. Also, we have that

$$U^* U = I \;\Leftrightarrow\; U U^* = I^* = I \,.$$

**Exercise 2.8** *Show that If $U_1, U_2$ are unitary, then so is $U_1 U_2$.*