REU 2013: Apprentice Program | **Summer 2013**

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1 Determinants continued

Recall the definition of the determinant,

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

where each $\sigma : [n] \to [n]$ is a permutation and $A \in M_n(F)$.

Claim 1.1 *If two columns of* A *are equal, then* $\det(A) = 0$

Proof: Let columns i and j be equal for the matrix A. Consider a permutation σ, which contributes the term $sgn(\sigma)a_{1,\sigma(1)}...a_{n,\sigma(n)}$. Consider another permutation, $\sigma' = \tau_{ij} \circ \sigma$, where τ_{ij} is the transposition of i and j. Since $sgn(\tau_{ij} \circ \sigma) = sgn(\tau_{ij}) \cdot sgn(\sigma) = -sgn(\sigma)$ and $a_{1,\sigma'(1)}...a_{n,\sigma'(n)} = a_{1,\sigma(1)}...a_{n,\sigma(n)}$, the two terms cancel.

Since the map $\sigma \mapsto \tau_{ij} \circ \sigma$ is a bijection from S_n to itself (which is its own inverse), for each permutation σ, the terms corresponding to σ and $\tau_{ij} \circ \sigma$ cancel each other out, and the determinant evaluates to 0. ■

Definition 1.2 (Elementary Column Operations) $A^{(j)} \leftarrow A^{(j)} - \lambda A^{(j')}$

Claim 1.3 *Elementary column operations do not change the determinant.*

Proof: Let A' be the matrix obtained via an elementary column operation $A^{(j)} \leftarrow A^{(j)} - \lambda A^{(j')}$ From the expansion we can see that $\det(A') = \det(A) + \det(A'')$ where A'' is the same as A except $(A'')^{(j)} = -\lambda A^{(j')}$. Also, $det(A'') = 0$ by Claim 1.1. ■

We can use elementary column(row) operations to transform a matrix into a triangular form such that the determinant is simply the product of the diagonal entries. We will prove some more basic facts about the determinant of a matrix.

Definition 1.4 (Transpose of a Matrix) $(A^T)_{ij} = A_{ji}$

Claim 1.5 $\det(A) = \det(A^T)$.

Proof:

$$\begin{aligned}
\det(A) &= \sum_{\sigma\in S_n} sgn(\sigma)\prod_{i=1}^{n} A_{i,\sigma(n)} = \sum_{\sigma\in S_n} sgn(\sigma)\prod_{i=1}^{n}(A^T)_{\sigma(i),i} \\
&= \sum_{\sigma\in S_n} sgn(\sigma)\prod_{i=1}^{n}(A^T)_{i,\sigma^{-1}(i)} \\
&= \sum_{\sigma^{-1}\in S_n} sgn(\sigma)\prod_{i=1}^{n}(A^T)_{i,\sigma^{-1}(i)} \\
&= \sum_{\sigma^{-1}\in S_n} sgn(\sigma^{-1})\prod_{i=1}^{n}(A^T)_{i,\sigma^{-1}(i)} \\
&= \det(A^T)\,.
\end{aligned}$$

The penultimate equality follows from noting that $sgn(\sigma) = sgn(\sigma^{-1})$, since $1 = sgn(\sigma \circ \sigma^{-1}) = sgn(\sigma)\cdot sgn(\sigma^{-1})$. ■

Example 1.6 *$det(A) \neq 0$ if and only if the columns of A are linearly indpendent.*

Proof: Assume the columns are not linearly independent. Then at least one of the columns is a linear combination of the others. Then one can obtain a matrix with a 0 column via elementary column operations, which must have determinant 0. Conversely, Gaussian elimination returns a full rank matrix if the columns are linearly independent, and the determinant, which is the product of the diagonal entries, is non-zero. ■

Definition 1.7 (Cofactors) *Let $A_{i,j}$ denote the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column of A. The cofaactor C_{ij} is defined as $C_{ij} = (-1)^{i+j}\cdot det(A_{i,j})$.*

Exercise 1.8 *Prove the cofactor expansion formula for the determinant showing that*

$$\det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\cdot\det(A_{i,j}) = \sum_{j=1}^{n} a_{ij}\cdot C_{ij}\,.$$

Exercise 1.9 *Evaluate the determinant of the following matrix, which has a's on the diagonal entries, and b's everywhere else.*

$$\begin{pmatrix} a & b & \dots & b & b \\ b & a & \dots & b & b \\ & & \dots & & \\ b & b & \dots & a & b \\ b & b & \dots & b & a \end{pmatrix}$$

Definition 1.10 (submatrix) *A submatrix of A is a matrix obtained from the entries in a subset of the columns and a (possibly different) subset of the rows of A.*

Exercise 1.11 *Show that rank*(A) *equals the largest* r *such that* A *has an* $r \times r$ *submatrix* A' *with* $\det(A') \neq 0$.

Exercise 1.12 *Show that* $det(AB) = det(A)det(B)$, *where* $A, B \in M_n(F)$.

Definition 1.13 *Let* $A \in F^{m \times n}$. $B \in F^{n \times m}$ *is called a left inverse if* $BA = I_n$. $C \in F^{m \times n}$ *is called a right inverse if* $AC = I_m$.

Exercise 1.14 *If* $m = n$ *then left inverse equals the right inverse.*

Exercise 1.15 *Show that if* $A \in M_n(F)$, *then* A^{-1} *exists if and only if* $\det(A) \neq 0$.

Definition 1.16 (Adjugate Matrix) *For a matrix* A, *the adjugate matrix of* A, *denoted* $adj(A)$, *is the matrix defined as*

$$(adj(A))_{ij} = C_{ji} = (-1)^{j+i} \det(A_{j,i}) \, .$$

The matrix $adj(A)$ *is sometimes also called the adjoint matrix, but since we will also use the term adjoint for a different matrix, we will refer to this one as the adjugate matrix.*

Exercise 1.17 *Show that* $A(adj(A)) = \det(A) \cdot I_n$

Note that the above shows that $A^{-1} = \frac{1}{\det(A)} \cdot adj(A)$.