REU 2013: Apprentice Program **Summer 2013**

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1 Linear Transformation continued

Recall that $\Phi : V \to W$ with $\overline{e} = \{e_1, \ldots, e_n\}$ as a basis for V and $\overline{f} = \{f_1, \ldots, f_m\}$ as basis for W. Then we can write M_Φ as following

$$M_\Phi = [\Phi]_{\overline{e},\overline{f}} = \begin{pmatrix} | & | & | \\ [\Phi(e_1)]_{\overline{f}} & \cdots & [\Phi(e_n)]_{\overline{f}} \\ | & | & | \end{pmatrix}$$

Example 1.1 *Let Φ be a rotation by angle θ, under usual basis. Then we can write the linear transformation as*

$$M_\Phi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Definition 1.2 (Composition of Linear Map) *Let $\Phi_1 : V \to W$ and $\Phi_2 : W \to Z$. Then $\Phi_2 \circ \Phi_1 : V \to Z$ where*

$$\Phi_2 \circ \Phi_1(v) = \Phi_2(\Phi_1(v))$$

Exercise 1.3

1. *Prove $[\Phi_2 \circ \Phi_1]_{\overline{e},\overline{z}} = [\Phi_2]_{\overline{f},\overline{z}} \cdot [\Phi_1]_{\overline{e},\overline{f}}$*
2. *Prove $rank(\Phi) = rank([\Phi]_{\overline{e},\overline{f}}$*

2 Determinant

Definition 2.1 (Permutation) *$\pi : [n] \to [n]$ is called* permutation *if π is a bijective map. The set of all permutations is denoted as S_n.*

Definition 2.2 (Inversion) *Inversion on a given permutation π is defined as*

$$Inv(\pi) = |\{(i, j) \mid i < j, \pi(i) > \pi(j)\}|$$

And also

$$sgn(\pi) = (-1)^{Inv(\pi)}$$

Exercise 2.3 *Let $\sigma, \pi \in S_n$, then $sgn(\sigma \circ \pi) = sgn(\sigma) \cdot sgn(\pi)$*

Claim 2.4

$$sgn(\pi) = \prod_{i<j} \left(\frac{\pi(i) - \pi(j)}{i - j} \right)$$

Proof: First, assume that we will get some number of absolute value 1. Then only the sign of each term matters (that is whether it is inverted or not). Therefore it will be exactly the same quantity as $sgn(\pi)$.

Now the absolute value should be 1, since all pairs (i, j) should appear on both numerator and denominator. ■

Definition 2.5 (Transposition) *Transposition is a swap of two elements.*

Exercise 2.6 *Prove that every permutation can be written as a composition of transpositions.*

Exercise 2.7 *If π is a transposition, then $sgn(\pi) = -1$*

Exercise 2.8 *Prove that every permutation can be written as a composition of "neighbor swaps" (transposition of $(i, i+1)$)*

Definition 2.9 (Determinant) *$A \in \mathbb{R}^{n\times n}$ or $A \in M_n(\mathbb{R})$. Then*

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

Example 2.10

$$det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right) = ad - bc$$

Exercise 2.11 *Prove that if a matrix A is a "triangular" matrix then $det(A)$ is the product of diagonal entries.*

Exercise 2.12 *Compute*

1. *$det(A)$ where $a_{i,i} = a$ and $a_{i,j} = b$ if $i \neq j$.*
2. *Determinant of Vandermonde matrix : $a_{i,j} = x_i^j$*
3. *$det(A)$ where $a_{i,i} = 1$, $a_{i,i+1} = -1$ and $a_{i+1,i} = 1$ and 0 otherwise.*

Proposition 2.13 *Let $A_{i,j}$ be a $(n-1) \times (n-1)$ matrix obtained by deleting ith row and jth column. Then*

$$det(A) = \sum_{j=1}^{n} a_{i,j}(-1)^{i+j} det(A_{i,j})$$

Exercise 2.14 *If two columns of A are equal, then $det(A) = 0$.*

Proposition 2.15 *Elementary operations do not change the determinant.*