REU 2013: Apprentice Program **Summer 2013**

Lecture 3: June 27, 2013

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1 Mapping between vector spaces

1.1 Basic definitions

Definition 1.1 (Linear Map) *We call a mapping* Φ *from vector space* V *to* W *a* **linear map** *if*

1. $\forall v_1, v_2 \in V, \quad \Phi(v_1 + v_2) = \Phi(v_1) + \Phi(v_2)$
2. $\forall \alpha \in \mathbb{F},\ \forall v \in V, \quad \Phi(\alpha \cdot v) = \alpha \cdot \Phi(v)$

Φ *is an* **isomorphism** *if it is a bijection.*

Exercise 1.2 *Show that* $\Phi(0) = 0$ *for any linear map* Φ*.*

Exercise 1.3 *Prove that an isomorphism maps an linearly independent set to an linearly independent set.*

Exercise 1.4 *Show that for an isomorphism* Φ*,* $rank(v_1, \ldots, v_n) = rank(\Phi(v_1), \ldots, \Phi(v_n))$*.*

Definition 1.5 (Kernel and Image) *Given* $\Phi : V \to W$*, the kernel and image of* Φ *are defined as*

$$ker(\Phi) := \{v \in V \ : \ \Phi(v) = 0\} \quad and \quad Im(\Phi) := \{\Phi(v) \ : \ v \in V\}.$$

Exercise 1.6 $ker(\Phi)$ *and* $Im(\Phi)$ *are vector spaces.*

The dimension of $(Im(\Phi)$ is called then rank of Φ and the dimension of $ker(\Phi)$ nullity of Φ.

Theorem 1.7 (Rank-Nullity Theorem) $dim(ker(\Phi)) + dim(Im(\Phi)) = dim(V)$*.*

Exercise 1.8 *Let* $V = \mathbb{R}^{\leq n}[x]$ *and* $\Phi = d/dx$*. What are* $ker(\Phi)$ *and* $Im(\Phi)$*?*

Proof of theorem: Let $dim(V) = n$, and $dim(ker(\Phi)) = k$. Let $v_1, \ldots, v_k$ be a basis for $ker(\Phi)$. Complete $v_1, \ldots, v_k$ to $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ which is a basis for V. We claim that $\Phi(v_{k+1}), \ldots, \Phi(v_n)$ form a basis for $Im(\Phi)$. We first prove that they are linearly independent.

Claim 1.9 $\Phi(v_{k+1}), \ldots, \Phi(v_n)$ *are linearly independent in* W

Proof of claim: Let $\alpha_{k+1} \cdot \Phi(v_{k+1}) + \cdots + \alpha_n$
$cdot \Phi(v_n) = 0$. Then $\Phi(\alpha_{k+1}v_{k+1} + \ldots + \alpha_n v_n) = 0$. This implies

$$\alpha_{k+1}v_{k+1} + \ldots + \alpha_n v_n \in ker(\Phi) = span(v_1, \ldots, v_k)\,.$$

However, since $v_1, \ldots, v_n$ are linearly independent, this gives that $\alpha_{k+1}, \ldots, \alpha_n = 0$. □

Now left to show that $Span(\Phi(v_{k+1}), \ldots, \Phi(v_n)) = Im(\Phi)$. Let $w \in Im(\Phi)$. Then $w = \Phi(v)$ for some $v \in V$. $w = \Phi(\alpha_1 v_1 + \ldots + \alpha_n v_n)$ for $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$.

$$\Phi(\alpha_1 v_1 + \ldots + \alpha_n v_n) = \underbrace{\alpha_1\Phi(v_1) + \ldots + \alpha_k\Phi(v_k)}_{=0} + \alpha_{k+1}\Phi(v_{k+1}) + \ldots + \alpha_n\Phi(v_n)$$

Therefore $Im(\Phi) \subseteq Span(\Phi(v_{k+1}), \ldots, \Phi(v_n))$. Also, for any $w \in Span(\Phi(v_{k+1}), \ldots, \Phi(v_n))$, of the form $w = \alpha_{k+1}\Phi(v_{k+1}) + \ldots + \alpha_n\Phi(v_n)$, we have that

$$w = \alpha_{k+1}\Phi(v_{k+1}) + \ldots + \alpha_n\Phi(v_n) = \Phi(\alpha_{k+1}v_{k+1} + \cdots + \alpha_n v_n)\,,$$

which shows that $Span(\Phi(v_{k+1}), \ldots, \Phi(v_n)) \subseteq Im(\Phi)$. ■

Exercise 1.10 *Let $b_1, \ldots, b_n$ be a basis for V. Then there is a unique linear map Φ such that $\Phi(b_1) = w_1, \ldots, \Phi(b_n) = w_n$*

Exercise 1.11

1. *An isomorphism maps a basis to a basis.*
2. *Any two vector spaces (over same set of scalars) with equal (finite) dimension are isomorphic (i.e. there exists an isomorphism between them)*

1.2 Linear map as a matrix

Proposition 1.12 *A $m \times n$ matrix A over $\mathbb{F}$ is a linear map from $\mathbb{F}^m$ to the column space of A*

Proof: $A(x_1 + x_2) = Ax_1 + Ax_2$ and $A(\alpha x) = \alpha Ax$. ■

Therefore, applying rank-nullity theorem to the matrix, since $Im(A)$ is the column-space of A and $ker(A) := \{x \mid Ax = 0\}$, $rank(A) + dim(ker(A)) = n$.

Let $\Phi : V \to W$ be a linear map. Let $\{e_1, \ldots, e_n\}$ be a basis for V, and $\{f_1, \ldots, f_m\}$ for W. Then given $v \in V$, $\exists$ unique $\alpha_1, \ldots, \alpha_n$ such that $v = \sum \alpha_i e_i$.

Let $[v]_{\overline{e}}$ denote $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Since for $\Phi(e_i) \in W$, $\exists \beta_{1i}, \ldots, \beta_{mi}$ such that $\Phi(e_i) = \beta_{1i}f_1 + \ldots + \beta mi f_m$.

Then we write

$$A_\Phi = \begin{pmatrix} | & | & | \\ [\Phi(e_1)]_{\overline{f}} & \cdots & [\Phi(e_n)]_{\overline{f}} \\ | & | & | \end{pmatrix} = \begin{pmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{m1} & \cdots & \beta_{mn} \end{pmatrix}$$

If we apply A_Φ to $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, we get

$$\begin{aligned} A_\Phi \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} &= \alpha_1 [\Phi(e_1)]_{\overline{f}} + \ldots + \alpha_n [\Phi(e_n)]_{\overline{f}} \\ &= [\Phi(\alpha_1 e_1 + \ldots + \alpha_n e_n)]_{\overline{f}} \end{aligned}$$

Thus if $[v]_{\overline{e}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, $A_\Phi [v]_{\overline{e}} = [\Phi(v)]_{\overline{f}}$

Exercise 1.13 *Write $\Phi = d/dx$ on $\mathbb{R}^{\leq n}[x]$ as a matrix.*

1.3 Application : Linear equations

Consider following system of linear equations.

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We can simply write it as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

We want to know when a system has a solution. This exactly corresponds to saying whether b is in $span(A^{(1)}, \ldots, A^{(n)})$.

Exercise 1.14 *$b \in span(A^{(1)}, \ldots, A^{(n)})$ if and only if $rank(A) = rank(A|b)$*

If $b = 0$, we call such system homogeneous system, and we ask $\exists ? x \neq 0$ such that $Ax = 0$.

Exercise 1.15 *Non-zero solution exists if and only if $rank(A) < n$.*

Exercise 1.16 (Graph coloring continued) *Show that the following graphs are 2-colorable (bipartite).*

1. $n \times m$ *grid*
2. *Hypercubes* ($V = \{0, 1\}^n$, $E = \{(x, y) \mid s$ and y differ in exactly one position$\}$)