UChicago REU 2013 Apprentice Program **Spring 2013**

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1 Extremal graph theory

Exercise 1.1 *What is the maximum possible number of edges of a graph on n vertices?*

(Answer: $\binom{n}{2} = n(n-1)/2 \sim n^2/2$)

Definition 1.2 *We say that two sequences $\{a_n\}$ and $\{b_n\}$ are* asymptotically equal*, denoted $a_n \sim b_n$, if $\lim_{n\to\infty} a_n/b_n = 1$.*

Exercise 1.3 *What is the number of edges of $K_{r,s}$, the complete bipartite graph? (Answer: rs.) What is the maximum possible number of edges of a bipartite graph on n vertices? (Answer: $\lfloor n^2/4 \rfloor$. Prove!)*

Exercise 1.4 (Mantel-Turán Theorem) ** If G is a triangle-free graph with n vertices and m edges then $m \le \frac{n^2}{4}$.*

Turán also solved the problem of the maximum number of edges when the graph does not contain K_q for a given q; the extremal graph is a complete $(q-1)$-partite graph with nearly equal parts.

Paul Turán (1910-1976), an analytic number theorist and close friend of Paul Erdős, also a mentor of your instructor, used to give commencement addresses in Budapest with the title "How to do good mathematics under the worst of circumstances." Turán proved now classical results in graph theory while serving in "labor battalions," doing in one case wiring on top of telephone poles – in complete privacy but with no paper or pencil. "Labor battalions" were military units set up by the Nazi-allied Hungarian regime where Jewish-Hungarian men were required to serve under harsh circumstances and were often treated with extreme cruelty by their Nazi commanders and guards. Few survived.

The following result provides a surprising contrast to the Mantel–Turán theorem: while excluding triangles only reduces the number of possible edges by a factor of 2, excluding 4-cycles reduces the order of magnitude of the possible number of edges.

Exercise 1.5 *(Kővári - Turán - Sós) If G has no C_4 (4-cycle), then $m \le cn^{3/2}$.*

Here c is a constant. Make your constant reasonably small. (Any constant $> 1/2$ works for sufficiently large n.)

(Note: Vera Sós is late Paul Turán's wife and also a mentor of your instructor.)

Given just the number of edges, m, but not the number of vertices of a graph, what would be a good strategy to construct a graph which would maximize the number t of triangles? Try to make each edge belong to as many triangles as possible. (If a graph has n vertices, an edge can belong to at most $n-2$ triangles. Why?) If possible, let's make it a complete graph and express the number triangles in terms of m.

Assume K_n has m edges. Let t be the number of triangles. $\Rightarrow m = \binom{n}{2}, t = \binom{n}{3}$ (combinatorial meaning?)

Then $2m = n(n-1) \sim n^2 \Rightarrow n \sim \sqrt{2m}$, $t = \frac{n(n-1)(n-2)}{6} \sim \frac{n^3}{6}$ (the $\sim$ sign refers to asymptotic equality, see above)

So for the complete graph K_n, $t \sim \frac{(2m)^{3/2}}{6} = \frac{\sqrt{2}}{3} m^{3/2}$.

Exercise 1.6 ** Prove: $\forall$ graph, $t \leq \frac{\sqrt{2}}{3} m^{3/2}$*

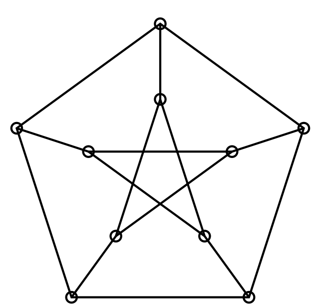
Hint: Spectral Theorem

2 Petersen's graph

Definition 2.1 (Bridge) *Given a connected graph G, a bridge is an edge whose removal disconnects the graph into two components.*

Exercise 2.2 ** Four Color Theorem $\Leftrightarrow$ Every bridgeless, 3-regular, planar graph has a proper edge coloring with 3 colors.*

A natural question to ask in such mathematical statements is to see if each condition is really necessary. In fact, Petersen was investigating the necessity of the condition of planarity in the above (right) statement when he discovered a counterexample - the Petersen graph, the single most famous graph in graph theory, an object of remarkable symmetry and mathematical beauty, comparable in ots uniqueness with the Platonic solids or Rubik's cube.



Petersen's graph

Exercise 2.3 ** Prove that the Petersen graph has no proper edge-coloring with 3 colors.*

Warning: I am not aware of an elegant proof of this statement. The Petersen graph has remarkable symmetry which can help cut down on the number of cases.

Definition 2.4 (Graph Isomorphism) *An isomporphism of the graphs G and H is a bijection $\varphi : V(G) \to V(H)$ that preseserves adjacency, i.e., a pair of vertices in G is adjacent iff the corresponding pair of vertices in H is adjacent.*

Definition 2.5 (Graph Automorphism) *An automorphism of the graph G is an isomorphism between G and itself, i.e., a permutation of the vertices that preserves adjacency.*

Notation: $\mathrm{Aut}(G)$ is the group of automorphisms of G.

Observation: $|\mathrm{Aut}(G)| \leq n!$.

Example 2.6 $|\mathrm{Aut}(Petersen)| = 120$*,* $|\mathrm{Aut}(dodecahedron)| = 120$*. However, the two are not isomorphic, as* $\mathrm{Aut}(Petersen) \cong S_5$*, but* $\mathrm{Aut}(dodecahedron) \ncong S_5$*. But they do share A_5 (the alternating group of degree 5, consisting of the even permutations of 5 elements). - Prove also that every pair of walks of length 3 in the Petersen graph are equivalent under automorphisms.*

3 Hoffman–Singleton Theorem

Exercise 3.1 ** Let $f(x) = x^4 + ax^3 + bx^2 + cx - 15 \quad (a, b, c \in \mathbb{Z})$. If $s \in \mathbb{Z}, f(s) = 0 \Rightarrow s \in \{$finite list$\}$. What is this list?*

Hint: $s^4 + as^3 + bs^2 + cs = 15$, where $a, b, c, s \in \mathbb{Z}$. What does this say about s and 15?

Recall the following exercise from yesterday's lecture.

Exercise 3.2 *If G is d-regular, girth $\geq 5 \Rightarrow n \geq d^2 + 1$.*

Proof: Pick a vertex x, which would have d neighbors. We need to show $\exists d^2 - d = d(d-1)$ additional vertices, or d groups of $d-1$ vertices. Consider each set of $d-1$ neighbors (excluding x) of the neighbors of x. Then no two can share a vertex, as this would create a C_4. Also, none has a vertex that is also a neighbor of x, as this would create a triangle. Thus, none of the $d(d-1)$ vertices repeats, and $n \geq d^2 + 1$. ■

A natural question that often leads to striking configurations is the case when equality holds: $n = d^2 + 1$. We observed that K_2 realizes equality for $d = 1$ ($n = 2$), and the pentagon (C_5) for $d = 2$ ($n = 5$). For $d = 3$, the Petersen graph gives the (unique) example with $n = 10$.

Equality is also attained for $d = 7, n = 50$, as demonstrated by the "Hoffman–Singleton graph," obtained by gluing together many copies of the Petersen graph in a devilishly clever way.

Theorem 3.3 (Hoffman–Singleton Theorem) *If G is d-regular, girth ≥ 5, $n = d^2 + 1 \Rightarrow d \in \{1, 2, 3, 7, 57\}$.*

We shall prove this remarkable result using the Spectral Theorem.

Proof: [Hoffman–Singleton Theorem]

Claim 3.4 *Every pair of non-adjacent vertices has exactly 1 common neighbor.*

Proof of claim. Let v, w be any non-adjacent pair of vertices in G. Let A be the set of neighbors of v, and let B be the set of neighbors of vertices in A (as in the proof of the inequality $n \geq d^2 + 1$). Since we assumed $n = d^2 + 1$, we must have $w \in B$, and hence the claim holds.

Claim 3.5 *Every pair of adjacent vertices in G has 0 common neighbors.*

Indeed, otherwise, we would have a triangle.

Note that these types of graphs are called *strongly-regular graphs*: regular graphs with two parameters r, s such that every pair of adjacent vertices has r common neighbors, and every pair of non-adjacent vertices has s common neighbors. The pentagon and the Petersen graph are strongly-regular graphs.

Let's look at the adjacency matrix $A = (a_{ij})$ of G. Then by definition, $\forall i, j \in [n]$

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{o.w.} \end{cases} \tag{1}$$

In particular, $\forall i \in [n], a_{ii} = 0$.

Let $B = A^2 = (b_{ij})$. What does b_{ij} represent in the graph G? (short, English phrase)
Answer: $b_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}$, where $a_{i\ell} b_{\ell j} = 1 \Leftrightarrow a_{i\ell} = 1, b_{\ell j} = 1 \Leftrightarrow i \sim \ell, \ell \sim j$, i.e., ℓ is a common neighbor of i and j, so b_{ij} counts the common neighbors of i and j. In particular, $b_i i = \deg(i)$.

In our case, by the previous two claims, we have $\forall i \in [n], b_{ii} = d$, and $\forall i \neq j$

$$b_{ij} = \begin{cases} 0 & \text{if } i \sim j \\ 1 & \text{if } i \nsim j \end{cases} \tag{2}$$

Comparing (1) and (2), we see that $\forall i \neq j, a_{ij} + b_{ij} = 1$. Therefore,

$$B + A = J + (d-1)I$$

where J is the $n \times n$ all 1's matrix. We conclude that

$$A^2 + A - (d-1)I = J. \tag{3}$$

Let's now look at the eigenvalues of A. Since G is d-regular, the largest eigenvalue of A is $\mu_1 = d$ with $\mathbb{1} = (1, \ldots, 1)^T$ as the eigenvector. Let $e_1 := \mathbb{1}$. Spectral Theorem tells us that we can extend any set of orthogonal eigenvectors of A to an orthogonal eigenbasis (we do not need them to be unit vectors, or normalized, for this proof).

So let $e_2, \ldots, e_n \in \mathbb{R}^n$ be orthgonal eigenvectors of A such that $e_2, \ldots, e_n \perp e_1$, and $\forall i, Ae_i = \mu_i e_i$ where $d = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

Claim 3.6 $(\forall j \geq 2)(Je_j = 0,\ A^2 e_j = \mu_j^2 e_j)$

$\rightarrow Je_j = (e_1 \cdot e_j, \ldots, e_1 \cdot e_j)^T = 0$. $A^2 e_j = A(\mu_j e_j) = \mu_j A e_j = \mu_j^2 e_j$.

Applying this to (3), for $j \geq 2$ we obtain

$$(A^2 + A - (d-1)I)e_j = \mu_j^2 e_j + \mu_j e_j - (d-1)e_j = Je_j = 0$$

But e_j is an eigenvector (non-zero), so $(\forall j \geq 2)(\mu_j^2 + \mu_j - (d-1) = 0)$ and each μ_j is a root of $t^2 + t - (d-1) = 0 \Rightarrow t = \frac{-1 \pm \sqrt{4d-3}}{2}$.

This tells us that A has at most 3 eigenvalues, $\lambda_0 = \mu_1 = d$, and $\lambda_{1,2} = \frac{-1 \pm \sqrt{4d-3}}{2}$, meaning that the last two eigenvalues have large multiplicities. Let's say that λ_1 has multiplicity m_1, and λ_2 has mutliplicity $m_2 \Rightarrow 1 + m_1 + m_2 = n = d^2 + 1 \Rightarrow m_1 + m_2 = d^2$.

Where else can we obtain a simple, easy-to-calculate linear equation of m_1, m_2 from A? $Tr(A)$.

$$Tr(A) = \sum_i a_{ii} = 0 = \sum_i \mu_i = d + m_1 \lambda_1 + m_2 \lambda_2$$

We now have the following two linear equations

$$m_1 + m_2 = d^2 \tag{4}$$
$$\lambda_1 m_1 + \lambda_2 m_2 = -d \tag{5}$$

Let $s = \sqrt{4d-3}$. Expanding out λ_1, λ_2 in (5), and then using (6), we get

$$-(m_1 + m_2) + s(m_1 - m_2) = -2d \Rightarrow s(m_1 - m_2) = d(d-2) \tag{7}$$

Case 1: $m_1 = m_2 \Rightarrow d = 2$

Case 2: $m_1 \neq m_2 \Rightarrow$ Since $d, m_1, m_2 \in \mathbb{Z}$, we must have $s \in \mathbb{Q}$ and therefore $s \in \mathbb{Z}$ (why?)

$$\begin{aligned} d^2 - 2d - s(m_1 - m_2) = 0 \\ \Rightarrow \left(\frac{s^2+3}{4}\right)^2 - 2\left(\frac{s^2+3}{4}\right) - s(m_1 - m_2) = 0 \\ \Rightarrow s^4 - 2s^2 - 16(m_1 - m_2)s - 15 = 0 \end{aligned}$$

We know from Exercise 3.1 that $s \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$.

$$d = \frac{s^2+3}{4} \in \{1, 3, 7, 57\}$$

$$\text{Case(1) \& Case(2)} \Rightarrow d \in \{1, 2, 3, 7, 57\}$$

■

This proof is one of the most wondrous mathematical gems known to the instructor. The existence of a graph for $d = 57$ remains an open problem; Michael Aschbacher has shown that if it exists, it cannot be quite as symmetrical as the smaller cases: not all vertices are equivalent under automorphisms.

4 Extremal set theory: Eventown, Oddtown

Recall that we are given n residents, and we are trying to maximize m, the number of clubs, under different club rules. Last time, we proved $m \leq n$ for Clubtown.

4.1 Eventown

Rules of Eventown:

- (Rule 0) all clubs are distinct $\Rightarrow m \leq 2^n$
- (Rule 1) (Eventown) all clubs must be even $\Rightarrow m \leq 2^{n-1}$
  Simple combinatorial proof of (# even subsets = # odd subsets of $[n]$?)
- (Rule 2) all pairs of clubs must share an even number of members.
  $\Rightarrow$ Married couple scheme (married couples join clubs together) gives $2^{\lfloor n/2 \rfloor}$.

Theorem 4.1 (Eventown Theorem, Elwyn Berlekamp) ** If there are n residents and m clubs in Eventown then $m \leq 2^{\lfloor n/2 \rfloor}$.*

The number of clubs in Eventown is still enormous, and the town officials could not possibly hope to keep even a list of all the clubs. Oddtown simply modifies (Rule 1) of Eventown by requiring that each club be odd (have an odd number of members).

4.2 Oddtown

Rules of Oddtown:

- (Rule 1) (Oddtown) all clubs must be odd
- (Rule 2) all pairs of clubs must share an even number of members.

(Rule 0) can be omitted because it follows from (Rule 1) and (Rule 2) of Oddtown (Why?).

Exercise 4.2 *Construct n Oddtown clubs.*

A solution: each resident forms a one-person club. It turns out that this is just one out of very many solutions.

Exercise 4.3 ** In Oddtown, there are $> 2^{\frac{n^2}{8}}$ ways of creating n clubs.*

Theorem 4.4 (Oddtown Theorem, Elwyn Berlekamp) ** If there are n residents and m clubs in Oddtown then $m \leq n$.*

Hint: Show that the incidence vectors of the clubs are linearly independent.