UChicago REU 2013 Apprentice Program	**Spring 2013**
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Exercise 0.1 ** If G is a regular graph of degree d and girth ≥ 5, then $n \geq d^2 + 1$.*

Definition 0.2 (Girth) *The girth of a graph is the length of the shortest cycle in it. If there are no cycles, the girth is infinite.*

Example 0.3 *The square grid has girth 4. The hexagonal grid (honeycomb) has girth 6. The pentagon has girth 5. Another graph of girth 5 is two pentagons sharing an edge. What is a very famous graph, discovered by the ancient Greeks, that has girth 5?*

Definition 0.4 (Platonic Solids) *Convex, regular polyhedra. There are 5 Platonic solids: tetrahedron (four faces), cube ("hexahedron" - six faces), octahedron (eight faces), dodecahedron (twelve faces), icosahedron (twenty faces). Which of these has girth 5?*

Can you model an octahedron out of a cube? Hint: a cube has 6 faces, 8 vertices, and 12 edges, while an octahedron has 8 faces, 6 vertices, 12 edges.

Definition 0.5 (Dual Graph) *The dual of a plane graph G is a graph that has vertices corresponding to each face of G, and an edge joining two neighboring faces for each edge in G.*

Check that the dual of an dodecahedron is an icosahedron, and vice versa.

Definition 0.6 (Tree) *A tree is a connected graph with no cycles.*

Girth(tree) $= \infty$. For all other connected graphs, Girth$(G) < \infty$.

Definition 0.7 (Bipartite Graph) *A graph is bipartite if it is colorable by 2 colors, or equivalently, if the vertices can be partitioned into two independent sets.*

Q. Given a (possibly huge) graph, what would be a simple evidence that it is not bipartite?
$\rightarrow$ An odd cycle.

Q. What would be a simple evidence that a graph is not 3-colorable?
$\rightarrow$ belief: there is no efficient way to do this; the shortest proof of non-3-colorability may be exponentially long.

In general, these kinds of theorems ($\exists$ a 2 coloring $\Leftrightarrow \nexists$ an odd cycle),

$$\exists \textit{something} \Leftrightarrow \nexists \textit{something else}$$

where both "things" are easily verifiable objects (like a 2-coloring or an odd cycle), are called *good characterizations.*

Example 0.8 *For some graphs, there are simple obstacles to to 3-colorability. E.g., the presence of a 4-clique (a K_4 subgraph), implies that the graph is not 3-colorable.*

Exercise 0.9 ** Find a graph which does not contain a K_4 yet is not 3-colorable.*

A pentagon is an example of a graph that does not contain a K_3, yet is not 2-colorable. Is there a way to add a single vertex to a pentagon such that it is not 3-colorable? Simply add a vertex and connect it with an edge to each of the 5 vertices.

Exercise 0.10 ** Trees are bipartite.*

Definition 0.11 (Chromatic Number) *The chromatic number of a graph $G, \chi(G)$ = minimum number of colors needed for coloring every vertex such that adjacent vertices get different colors.*

So G is k-colorable iff $\chi(G) \leq k$ ("k colors suffice").

Definition 0.12 (Planar Graph) *A graph G is planar if it has ($\exists$) a plane drawing - a drawing of the graph in the plane such that no edges intersect.*

Example 0.13 *The graphs fromed by the edges of the Platonic solids are planar. In general, if one can draw a graph on the sphere, then it can be drawn in the plane as well. What are some examples of non-planar graphs?*

Definition 0.14 (Complete Bipartite Graph) *A bipartite graph where every "red" vertex adjacent to every "blue" vertex. $K_{r,s}$ denotes a complete bipartite graph with r "red" vertices and s "blue" vertices.*

Theorem 0.15 *K_5 and $K_{3,3}$ are non-planar.*

Definition 0.16 (Topological K_5) *Start with a K_5 and split add a new vertex on an edge (so now we have 6 vertices and 11 edges, but the graph still "looks" the same). Repeat the process any number of times, adding new vertices on any edges. A graph obtained by this process is called a "topological K_5." (Likewise for topological $K_{3,3}$.)*

Theorem 0.17 (Kuratowski's Theorem) *G is not planar $\Leftrightarrow$ G contains a topological $K_{3,3}$ or a topological K_5 (Kuratowski subgraphs).*

Note that this is an example of a good characterization. It is not hard to prove that every planar graph is 6-colorable. It was proven about 100 years ago that every planar graph is 5-colorable. The *4-color conjecture*, that every planar graph is 4-colorable, was the most famous open problems in graph theory, until it was proven in 1974 by mathematicians Appel and Haken at UIUC, with the aid of computers. ("Four colors suffice" postmarks were issued by UIUC on the occasion.)

Exercise 0.18 (Grötzsch's Graph) ** Find a graph G such that* $\chi(G) \geq 4$ *and* $G \not\supseteq K_3$ *(G is triangle-free). Hint: 11 vertices, 5-fold symmetry (invariant to rotations by* $2\pi/5$*)*

Exercise 0.19 *** $(\forall k)(\exists G)(G \not\supseteq K_3$ *and* $\chi(G) \geq k)$.

Theorem 0.20 (Erdős) $(\forall k, g)(\exists G)(girth(G) \geq g)$ *and* $\chi(G) \geq k)$.

(Back to the first exercise of the lecture) If G is d-regular, girth $\geq 5 \Rightarrow n \geq d^2 + 1$

Q. Can we have $n = d^2 + 1$? Try for $d = 1, 2, 3$
$d = 1$ (an edge)
$d = 2$ (pentagon)
$d = 3$ (Petersen's graph)

Theorem 0.21 (Hoffman-Singleton) *If G is a regular graph of degree d and girth* ≥ 5 *and* $n = d^2 + 1$ *then* $d \in \{1, 2, 3, 7, 57\}$.

Exercise 0.22 ** Let* $f, g \in \mathbb{Z}[x]$ *(set of polynomials with integer coefficients). Decide whether or not* f, g *have a common complex root, using rational numbers only.*

Exercise 0.23 (Clubtown) *We have a town of n residents who love to form clubs. Rule: every pair of clubs shares exactly 1 member. How many clubs can this town have? (Assume there are no two clubs have identical membership.)*

Theorem 0.24 (Clubtown Theorem - Erdős-DeBruijn) *The number of clubs in Clubtown is* $\leq n$.

First solve the easy case when there is a one-person club. Let us assume from now on that every club has more than one member.

The original proof of the theorem was purely combinatorial. We will prove this by transforming the problem into a problem in linear algebra. This idea is due to R. C. Bose (1950) and it started the highly successful linear algebra method in combinatorics.

Definition 0.25 (Incidence Vector) *Let* $S \subseteq [n] = \{1, ..., n\}$. *An incidence vector* $\mathbb{1}_S \in \mathbb{R}^n$ *is defined as*

$$(\mathbb{1}_S)_j = \begin{cases} 1 & \text{if } j \in S \\ 0 & o/w \end{cases}$$

Exercise 0.26 *Given* $S, T \subseteq [n]$ *we have* $|S \cap T| = \langle \mathbb{1}_S, \mathbb{1}_T \rangle$.

Proof of Theorem (outline):
Assume that we have n residents, and a set system $\mathcal{S} = \{S_1, ..., S_m\}$ s.t. $S_j \in [n]$ for each club. Let the incidence vectors of the clubs be $v_1, ..., v_m$. The condition that every pair of clubs shares

exactly 1 member is $(\forall i \neq j)(v_i \cdot v_j = 1)$. As stated above, we may also assume that $(\forall i)(v_i \cdot v_i > 1)$.

We will prove: $v_1, ..., v_m$ are linearly independent. Then by the 1st Miracle of Linear Algebra, we will have $m \leq n$, which is our desired conclusion.

Definition 0.27 (Incidence Matrix of a Set Sytem) *M, the incidence matrix of a set system $\mathcal{S} = \{S_1, ..., S_m\}$, each $S_i \in [n]$, is the $m \times n$ matrix obtained by setting v_i, the incidence vector of S_i as the i'th row of M.*

The clubtown rule, $(\forall i \neq j)(v_i \cdot v_j = 1)$ can be expressed as

$$MM^T = \begin{pmatrix} a_{1,1} & & \mathbf{1} \\ & \ddots & \\ \mathbf{1} & & a_{n,n} \end{pmatrix}$$

where the diagonal entries $a_{ii} = v_i \cdot v_i = |S_i \cap S_i| = |S_i| > 1$, and $(\forall i \neq j)((MM^T)_{ij} = 1)$. In terms of M, we are trying to show that $rk(M) = m$ (full rank - the rows, $v_1, ..., v_m$ are linearly independent).

Exercise 0.28 ** If $M \in \mathbb{R}^{k \times l} \Rightarrow rk(M) = rk(MM^T)$.*

We shall show that $rk(M) = m$ by showing $rk(MM^T) = m$. (This does not require the exercise above; we only need the simpler fact, already discussed, that $rk(A) \geq rk(AB)$.)
Recall that for an $n \times n$ matrix A, the following are equivalent:
1. $rk(A) = n$
2. the rows of A are linearly independent
3. the rows of A span $\mathbb{R}^n$
4. the columns of A are linearly independent
5. the columns of A span $\mathbb{R}^n$
6. A has a left inverse
7. A has a right inverse
8. A has an inverse
9. $\det(A) \neq 0$

Such matrices are called **non-singular**.

Exercise 0.29 *(a's on the diagonal, b's everywhere else)*

$$\det \begin{pmatrix} a & & \boldsymbol{b} \\ & \ddots & \\ \boldsymbol{b} & & a \end{pmatrix} = (a + (n-1)b)(a-b)^{n-1}$$

Hint: perform elementary column operations to make the last column all 1's (or the same value), then remove all b's to create an upper-triangular matrix.

Proof: [Clubtown Theorem for r-uniform Set System]
Suppose M is the incidence matrix of Clubtown with n residents and m clubs. We want to show that $M^T M$ is non-singular, as this would imply M has rank m.

Let's assume for now that our clubs form an *r-uniform set system*: $(\forall i)(|S_i| = r)$: all clubs have the same number of members. We can then use the previous exercise to show that $\det(M^T M) = (r + (m-1))(r-1)^{m-1} \neq 0$ (because $r > 1$). ■

What if the diagonal entries are not necessarily equal (the clubs may not have uniform size)? One can still calculate the determinant of $M^T M$, as done in 1956 by Majumdar. We shall follow a more elegant path that requires virtually no calculation.

Exercise 0.30 *A symmetric real matrix is positive definite if and only if it is positive semidefinite and nonsingular.*

(Hint: the product of the eigenvalues is the determinant)

Exercise 0.31 *Let J be the square matrix with all entries 1. Then J is positive-semidefinite.*

Proof: $x^T J x = (\sum x_i)^2$.

Claim 0.32

$$A = \begin{pmatrix} a_{1,1} & & \mathit{1} \\ & \ddots & \\ \mathit{1} & & a_{n,n} \end{pmatrix}$$

If $(\forall i)(a_{ii} > 1)$ then A is positive definite.

Proof: We have $A = J + D$, where D is diagonal with $D_{ii} = a_{ii} - 1$. We saw that J is positive semidefinite and D is positive definite, and thus A is positive definite (why?). ■

Proof: [Clubtown Theorem]
Let M be the incidence matrix of Clubtown with n residents and m clubs. Then $M^T M$ is positive-definite by the previous exercise and therefore it is non-singular. So M has rank m, as desired. ■

We can generalize the problem further, by requiring that every pair of clubs share the same number $t \geq 1$ members. The same result holds:

Theorem 0.33 (Generalized Fisher Inequality) *Consider a system $S_1, \ldots, S_m$ of m subsets of a set of n elements satisfying the condition that $(\forall i \neq j)(|S_i \cap S_j| = t)$. Then $m \leq n$.*

Hint: use the same method as above.

There is no purely combinatorial proof known for this theorem. All known proofs are variations of the linear algebra proof above.