UChicago REU 2013 Apprentice Program	**Spring 2013**

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The students asked the following questions: #69 (Annie), Oddtown Problem (Nate), #78 (Annie, Philip), #82(3) (Freddy), and (*) $\lambda = 0$ has multiplicity $k \Leftrightarrow G$ has k connected components (Annie). We started with problem #82(3). New problems are marked with a *.

Definition 0.1 (bilinear form, quadratic form) *Let $A \in M_n(\mathbb{R})$. The* **bilinear form** *associated with A is a function $B_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that*

$$B_A(x, y) = x^T Ay \quad (x, y \in \mathbb{R}^n)$$

Note that the dot product is a special case of a bilinear form, where $A = I$.
Let us compute B_A explicitly: let $A = (a_{ij}), x = (x_1, \dots, x_n)^T, y = (y_1, \dots, y_n)^T$. Then

$$B_A(x, y) = x^T Ay = \sum_i \sum_j a_{ij} x_i y_j. \tag{1}$$

The **quadratic form** associated with A is a function $Q_A : \mathbb{R}^n \to \mathbb{R}$ defined as

$$Q_A(x) = B_A(x, x) = x^T Ax = \sum_i \sum_j a_{ij} x_i x_j. \tag{2}$$

Exercise 0.2 $(\forall A \in \mathbb{R}^{n\times n})(\exists!$ *a symmetric matrix* $A')(Q_A(x) = Q_{A'}(x)$.
Note: $\exists!$ means "exists unique . . . such that"

Let's look at an example of the quadratic form of $A \in \mathbb{R}^{2\times 2}$.

Example 0.3 *Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix}, x = (x_1, x_2)^T$. What is A'?*

$$Q_A(x) = 2x_1^2 + 3x_1x_2 - x_2x_1 + 7x_2^2 = 2x_1^2 + 2x_1x_2 + 7x_2^2 = Q_{A'}(x)$$

Answer: $A' = \begin{pmatrix} 2 & 1 \\ 1 & 7 \end{pmatrix}$

Example 0.4 *Let $D = \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}$. If we set $Q_D(x) = 8x_1^2 + 3x_2^2 = 1$, this is the equation for an ellipse in $\mathbb{R}^2$, $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, where $a = \frac{1}{\sqrt{8}}, b = \frac{1}{\sqrt{3}}$.*

If we set $Q_A(x) = 2x_1^2 + 2x_1x_2 + 7x_2^2 = 1$ in the previous example, this also represents an ellipse. Can you find the vectors (directions) for the major and minor axes of this ellipse?

Definition 0.5 (Orthogonal Matrix) *Recall that $M \in M_n(\mathbb{R})$ is orthogonal if $M^T M = I$. If $M = [c_1, \ldots, c_n]$, then this means $c_i^T c_j = \delta_{ij}$, the kronecker delta function, defined as*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So the columns of an orthogonal matrix form an orthonormal basis of $\mathbb{R}^n$, and the converse is also true.

Claim: the rows of an orthogonal matrix form an orthonormal basis (The 3rd Miracle of Linear Algebra). If $M^T M = I$, then M^T is the left-inverse of M. Can we say anything about the right inverse of M? We prove the following exercises.

Exercise 0.6 *If an $n \times n$ matrix has a left inverse, it has a right inverse.*

Exercise 0.7 *(Problem 33) Let B be a $k \times n$ matrix. When does B have a left inverse?*

Review of The 2nd Miracle of Linear Algebra

Recall that the most important parameter of a matrix (after its dimensions) is its rank. The 2nd Miracle of Linear Algebra says that row rank = column rank. We prove the 2nd miracle by performing elementary column operations: $c_j \leftarrow c_i + \lambda c_j$. We need to show

1. elementary column operations do not change the column rank
2. elementary column operations do not change the row rank.

Recall that given $B = [c_1, \ldots, c_n]$, $col(B) = span(c_1, \ldots, c_n)$ and $\dim(col(B)) = col\_rank(B)$ (by The 1st Miracle of Linear Algebra). Since the column space does not change from elementary column operations, $col\_rank(B') = col\_rank(B)$, where B' is the resulting matrix after applying an elementary column operation to B.

Exercise 0.8 *Find an example where an elementary column operation changes the row space.*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

This shows that the column space is invariant to elementary column operations while the row space is not. Nevertheless, the row rank is invariant to elementary column operations. In fact, all linear relations among the rows are invariant to elementary column operations. A *linear relation* among the vectors $r_1, \ldots, r_k$ is an equation of the form $\alpha_1 r_1 + \cdots + \alpha_k r_k = 0$. If a linear relation holds for a set of rows before an elementary column operation, it holds for the same rows after, with the same coefficients. (Try this)

After all the elementary column operations, we can have the final matrix in the form $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, which shows the 2nd miracle.

Exercise 0.9 ** Prove:* $rank(AB) \leq min(rk(A), rk(B))$

Note that $col(A) = \{Ax : x \in \mathbb{R}^n\}$. Then $AB = A[b_1, \ldots, b_m] = [Ab_1, \ldots, Ab_m] \Rightarrow col(A) \supseteq col(AB)$. Therefore, $rk(A) = \dim(col(A)) \geq \dim(col(AB)) = rk(AB)$.

Likewise, $rk(AB) \leq rk(B)$, which can be proved by repeating the previous argument with row spaces of AB and B. there is a half-line lemma (a mathematical statement used to help prove a theorem - sometimes lemmas are more important than theorems, as they can be used to prove many more theorems) that can be used.

Exercise 0.10 ** Lemma:* $(AB)^T = B^T A^T$

(Back to Exercise 7)

Definition 0.11 *C is a left inverse of B if $CB = I_n$ where I_n is the $n \times n$ identity matrix.*

For such a C to exist, it is necessary that $rk(B) \geq n$, i.e. $rk(B) = n$ (since B is $k \times n$), i.e. the columns of B are linearly independent. So A has a left inverse $\Leftrightarrow$ A has full column rank. A has a right inverse $\Leftrightarrow$ A has full row rank.

Exercise 0.12 ** For a square matrix, if $\exists$ a left inverse and $\exists$ a right inverse $\Rightarrow$ they are the same. (The proof is half a line.)*

Going back to orthogonal matrices, we now have that if M is square, then $M^T M = I$ gives us $M^T = M_{left}^{-1} = M_{right}^{-1}$.

Recall the spectral theorem for real, symmetric matrices.

Theorem 0.13 (Spectral Theorem) *If $A \in M_n\mathbb{R}$, and A is symmetric, $A^T = A$, then $\exists C$, orthogonal, such that $C^T AC = D$, where D is diagonal.*

Another way of stating the spectral theorem: If $A \in M_n(\mathbb{R})$ is symmetric $\Rightarrow$ $\exists$ an orthonormal eigenbasis of A. (i.e. an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A)

If we look at the quadratic form associated with a real, symmetric matrix A,

$$x^T Ax = y^T Dy \quad (D = C^T AC,\ y = C^{-1}x)$$

(Back to Finding the major/minor axes of an ellipse) So in general, by the previous exercise from the beginning of this problem session, given any matrix $A \in M_n(\mathbb{R})$, we have A', symmetric, which gives $Q_A(x) = Q_{A'}(x) = x^T A'x = y^T Dy$, where $D = C^T A'C$. Then $y^T Dy = 1$ gives the equation of an ellipsoid (in $\mathbb{R}^n$), in terms of the eigenvalues (diagonal entries of D) and the coordinates of the the new eigenbasis.

For example, if we have $A = \begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix}$ such that $x^T Ax = 2x_1^2 + 2x_1x_2 + 7x_2^2$, we find $A' = \begin{pmatrix} 2 & 1 \\ 1 & 7 \end{pmatrix}$, which is diagonalizable by the spectral theorem. Then $2x_1^2 + 2x_1x_2 + 7x_2^2 = \lambda_1 y_1^2 + \lambda_2 y_2^2$, where λ_1, λ_2 are the diagonal entries of $D \sim A'$, which are also the eigenvalues of A'. The axes of the ellipse would have directions corresponding to the eigenvectors of A'.

Exercise 0.14 $trace(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$

Exercise 0.15 ** An eigendirection is a subspace spanned by a single eigenvector. Which $n \times n$ real, symmetric matrices have more than n eigendirections? If so, how many?*

For example, the $n \times n$ identity matrix has infinitely many eigendirections.

Definition 0.16 (Positive Semi-definite) *A symmetric matrix $A \in M_n(\mathbb{R})$ is positive semi-definite if* $(\forall x \in \mathbb{R}^n)(x^T Ax \geq 0)$.

Definition 0.17 (Positive Definite) *A symmetric matrix $A \in M_n(\mathbb{R})$ is positive definite if* $(\forall x \in \mathbb{R}^n, x \neq 0)(x^T Ax > 0)$.

Exercise 0.18 *Let A be a symmetric real matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove:*
(a) A is positive definite if and only if $(\forall i)(\lambda_i > 0)$.
(b) A is positive semi-definite if and only if $(\forall i)(\lambda_i \geq 0)$.

Exercise 0.19 *Problem 82(3). A is positive semi-definite $\Leftrightarrow (\exists B)(A = B^T B)$*

Assume $A = B^T B$ (B can be rectangular). Then $x^T Ax = x^T B^T Bx = ||Bx||^2 \geq 0$.
Conversely, let A be positive semi-definite. We can make a stronger claim with this assumption.

Exercise 0.20 (Square Root of Positive Semidefinite Matrix) *A is positive semi-definite $\Rightarrow$ $\exists! B$ such that $A = B^2$, $B = B^T$, and B is positive semi-definite. We call this matrix B the "square root of A" ($\sqrt{A}$).*