Recap

Pinsker's inequality: $D(P \| Q) \geqslant \frac{1}{2 \ln 2} \| P - Q \|_1^2$

Continuous random variables: Prob. density function $p(x)$

s.t. $\mathbb{P}[x \in B] = \int_B p(x)\, dx$

Gaussian r.v.s $p(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}$ for 1-d

$p(x) = \frac{e^{-(x-\mu)^T \Sigma^{-1} (x-\mu)/2}}{(2\pi)^{n/2} \cdot (\det(\Sigma))^{1/2}}$ for n-dimensions

Differential entropy

$$h(X) = \int_x p(x) \cdot \log \frac{1}{p(x)} \, dx$$

e.g.

X uniform on $[0,1]$: $h(X) = \int 1 \log 1 \, dx = 0$

$p(x) = 1$

$Y = \frac{X}{2}$: $h(Y) = \int 2 \log \frac{1}{2} \, dx = -1$

$p(y) = 2$

$Z = X^2$: $h(Z) = \int \frac{1}{2\sqrt{z}} \log 2\sqrt{z} = 1 - \frac{1}{\ln 2}$

$p(z) = \frac{1}{2\sqrt{z}}$

KL-divergence to the rescue

Two distributions with densities
$p(x)$, $q(x)$

$$D(P \,\|\, Q) = \int_{\mathcal{X}} p(x) \log \frac{p(x)}{q(x)} \, dx$$

▹ For $Y : \mathcal{X} \longrightarrow [n]$ (measurable) function of x

$$D(P(Y) \,\|\, Q(Y)) \leq D(P(X) \,\|\, Q(X))$$

Can show $D(P(X) \,\|\, Q(X)) = \sup_{Y, n} \underbrace{D(P(Y) \,\|\, Q(Y))}_{\text{finite case}}$

Rescuing what we can

Mutual information: $\int p(x,y) \log \frac{p(x|y)}{p(x)}$

$$= D\left(p(X,Y) \,\|\, p(X)\cdot p(Y)\right)$$

$$\overset{?}{=} h(X) - h(X|\underline{Y})$$

Inequalities:

- Jensen's ✓
- $h(X|Y) \le h(X)$ ✓
- Subadditivity ✓
- Pinsker ✓

Entropy Computations: Changing variables

- For $Y = a \cdot X$ $(a \in \mathbb{R})$, $h(Y) = h(X) + \log|a|$ $\quad a \neq 0$, say $a > 0$

Say $f(x)$ = density of X, $g(y)$ = density of Y

$$\mathbb{P}[Y \in [b, c]] = \mathbb{P}\left[X \in \left[\frac{b}{a}, \frac{c}{a}\right]\right] = \int_{b/a}^{c/a} f(x)\, dx$$

$$\parallel$$

$$\int_b^c g(y)\, dy \qquad = \int_b^c f\left(\frac{y}{a}\right) \cdot \frac{1}{a} \cdot dy \qquad \left(y = \frac{x}{a}\right)$$

$$\therefore g(y) = \frac{1}{a} f\left(\frac{y}{a}\right). \quad h(Y) = \int \frac{1}{a} f\left(\frac{y}{a}\right) \log \frac{a}{f(y/a)}\, dy = \int f(x) \log \frac{a}{f(x)}\, dx$$

$$= h(X) + \log a$$

- For $Y = X + c$, $h(Y) = h(X)$

$$g(y) = f(y-c)$$

$$h(Y) = \int f(y-c) \log \frac{1}{f(y-c)} = \int f(x) \log \frac{1}{f(x)} = h(X)$$

Entropy Computations: Scaling vectors

- For $\text{Supp}(X) \subseteq \mathbb{R}^n$, non-singular $A \in \mathbb{R}^{n \times n}$, $\underline{Y} = AX$

$$h(\underline{Y}) = h(X) + \log|\det(A)|$$

$$g(y) = \frac{1}{|\det(A)|} f(A^{-1}y)$$

Gaussian entropy: one dimension

- $X \sim N(\mu, \sigma^2)$. $\quad p(x) = \dfrac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}$

$\mathbb{E}[X] = \mu$

$\mathbb{E}[(X-\mu)^2] = \sigma^2$

$$h(X) = \int_{\mathbb{R}} p(x) \log \frac{1}{p(x)}\, dx$$

$$= \mathop{\mathbb{E}}_{X\sim p} \log \frac{1}{p(X)}$$

$$= \frac{1}{\ln 2} \mathop{\mathbb{E}}_{X\sim p}\left[\frac{(X-\mu)^2}{2\sigma^2} + \ln(\sqrt{2\pi}\,\sigma) \right]$$

$$= \frac{1}{\ln 2}\left(\frac{1}{2} + \frac{1}{2} \ln(2\pi\sigma^2) \right)$$

$$= \frac{1}{2} \log(2\pi\sigma^2 \cdot e)$$

Gaussian entropy: n dimensions

- $X \sim N(0, I_n)$ $\quad X = (X_1 \dots X_n)$ where each $X_i \sim N(0,1)$ i.i.d.

$$h(X) = \frac{n}{2}\log(2\pi e)$$

- $X \sim N(\mu, \Sigma)$

$$h(X) = \frac{n}{2}\log(2\pi e) + \frac{1}{2}\log(\det(\Sigma))$$

$$X = \Sigma^{1/2} Y$$

Gaussian KL-divergence: one dimension

$P = N(\mu_1, \sigma_1^2)$. $Q = N(\mu_2, \sigma_2^2)$

$$D(P \,\|\, Q) = \int_{\mathbb{R}} p(x) \log\left(\frac{p(x)}{q(x)}\right)$$

$$= \mathop{\mathbb{E}}_{x \sim P} \frac{1}{\ln 2} \cdot \ln\left(\frac{e^{-(x-\mu_1)^2/2\sigma_1^2}}{\sqrt{2\cancel{\pi}}\,\sigma_1} \cdot \frac{\sqrt{2\cancel{\pi}}\,\sigma_2}{e^{-(x-\mu_2)^2/2\sigma_2^2}} \right)$$

$$= \frac{1}{\ln 2} \mathop{\mathbb{E}}_{x \sim P}\left[-\frac{(x-\mu_1)^2}{2\sigma_1^2} + \frac{(x-\mu_2)^2}{2\sigma_2^2} + \ln\frac{\sigma_2}{\sigma_1} \right]$$

$\mathbb{E}(x-\mu_1)^2 = \sigma_1^2$ $\quad$ $\mathbb{E}\left[(x-\mu_1+\mu_1-\mu_2)^2\right]$

$= \sigma_1^2 + (\mu_1-\mu_2)^2$

$$= \frac{1}{\ln 2}\left[\frac{\sigma_1^2 - \sigma_2^2 + (\mu_1-\mu_2)^2}{2\sigma_2^2} + \ln\frac{\sigma_2}{\sigma_1} \right]$$

<u>Ex</u>: $D(P \,\|\, Q)$ for $P = N(\mu_1, \Sigma_1)$ and $Q = N(\mu_2, \Sigma_2)$

Perturbation bounds

- $P = N(\mu_1, \sigma^2)$, $Q = N(\mu_2, \sigma^2)$

$$\|P - Q\|_1 \leq ?$$

$$D(P \| Q) = \frac{1}{\ln 2} \cdot \frac{(\mu_1 - \mu_2)^2}{2\sigma^2}$$

$$\therefore \|P - Q\|_1 \leq \sqrt{2 \ln 2 \cdot D(P \| Q)} = \frac{|\mu_1 - \mu_2|}{\sigma}$$