HW 3 posted

Due: 2/21/25

Recap

I-Projections: Closed, convex Π

$$P^* = \underset{P \in \Pi}{\operatorname{argmin}}\, D(P \| Q) = \mathrm{Proj}_{\Pi}(Q)$$

Defect inequality: $D(P \| Q) \geq D(P \| P^*) + D(P^* \| Q)$

Linear families: $\mathcal{L} = \{ P \mid \mathbb{E}_{X \sim P} f_i(X) = \alpha_i \quad \forall i \in [k] \}$

$$D(P \| Q) = D(P \| P^*) + D(P^* \| Q)$$

$$P^*(x) = c \cdot Q(x) \cdot e^{\sum \lambda_i f_i(x)}$$

[Wigderson]
Operator Scaling

Matrix Scaling (Sinkhorn)

- $M \in \mathbb{R}_+^{n \times n}$ $\quad M_{ij} > 0 \quad \forall i, j$

There exist diagonal matrices D_1, D_2 s.t. $M' = D_1 M D_2$ is doubly stochastic

i.e.

$$\forall i \sum_j M'_{ij} = \frac{1}{n} \qquad \forall j \sum_i M'_{ij} = \frac{1}{n}$$

<u>Proof:</u>

$M \simeq Q$

doubly stochastic $\simeq \mathcal{L}$

$B = \sum M_{ij}$

$$M'' = \frac{M}{B}$$

Form of the optimizer

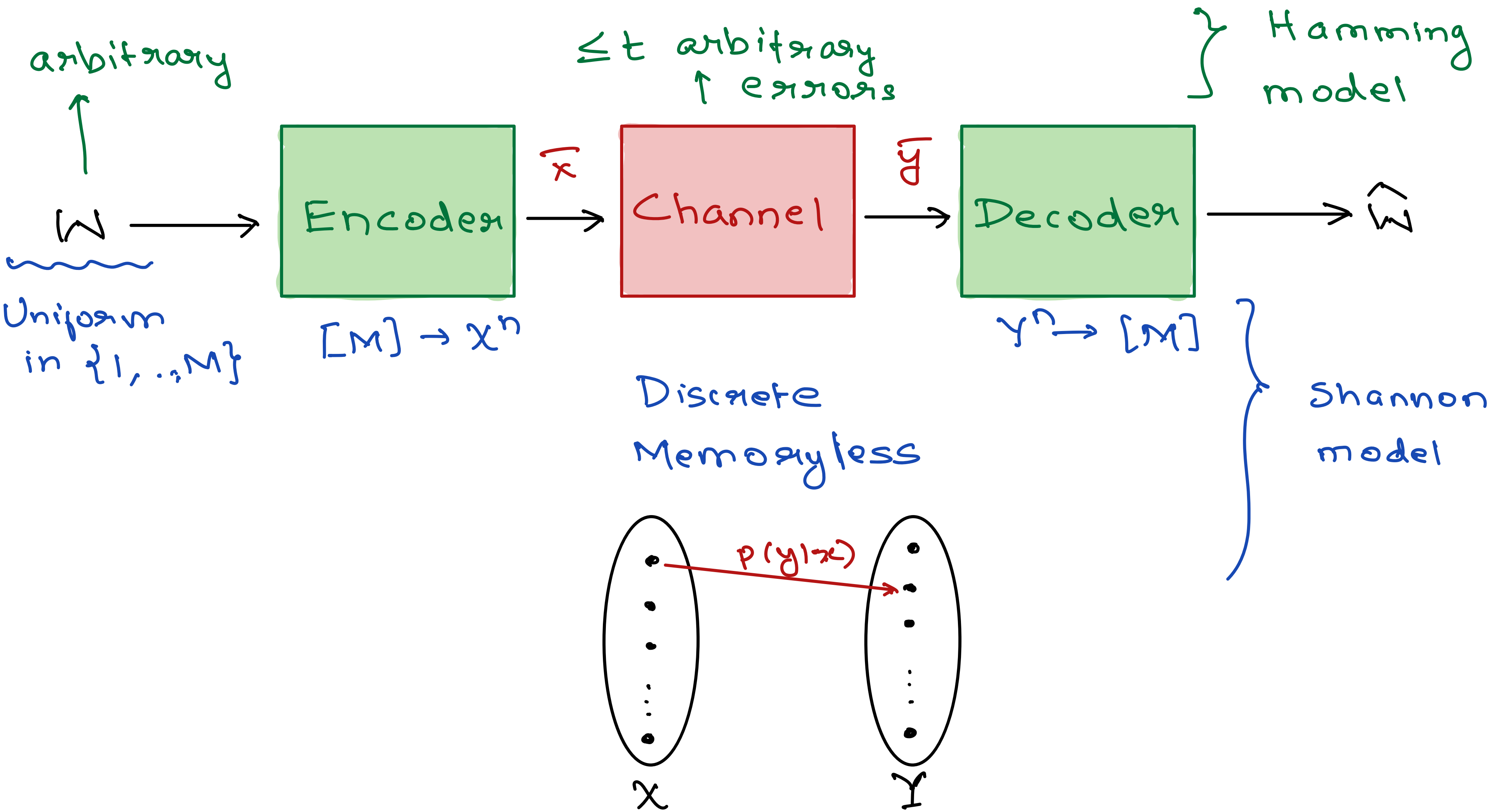
$$P^*(i,j) = c_0 \cdot Q(i,j) \cdot \exp\left(\sum_{i_0} \lambda_{i_0} f_{i_0}(i,j) + \sum_{j_0} \mu_{j_0} g_{j_0}(i,j)\right)$$

$$P^*(i,j) = c_0 \cdot M'_{ij} \cdot e^{\lambda_i + \mu_j}$$

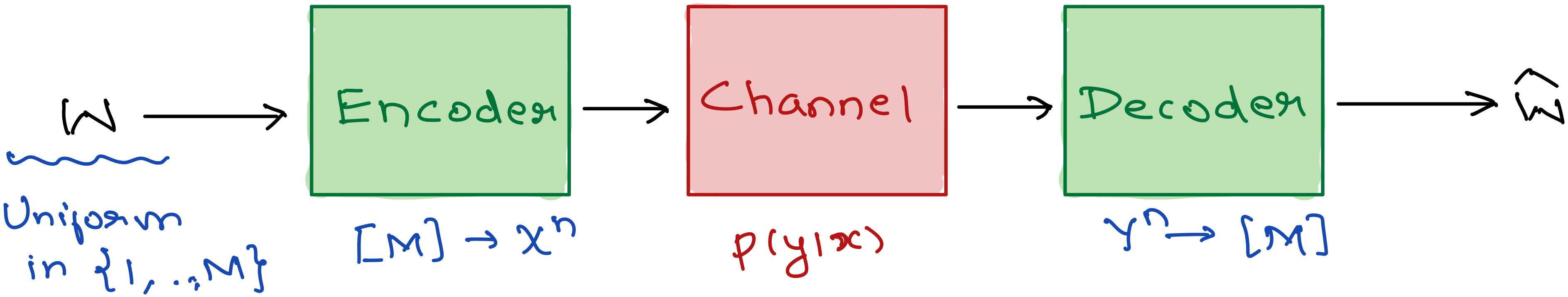
$$= \underbrace{\sqrt{c_0} \cdot e^{\lambda_i}}_{d_i} \cdot M''_{ij} \underbrace{\sqrt{c_0}\, e^{\mu_j}}_{d_j'}$$

$$= (D_1 M'' D_2)_{ij}$$

Error Correcting Codes



The Shannon model



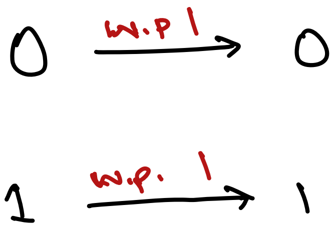
Rate: $R = \dfrac{\log M}{\log |\mathcal{X}|^n} = \dfrac{\log M}{n \cdot \log |\mathcal{X}|}$

Capacity: $C = \max\limits_{P(x)} I(X;Y)$

Channels and Capacities ($\mathcal{X} = \{0,1\}$)

$$C = \max_{p(x)} I(X;Y)$$

Noiseless

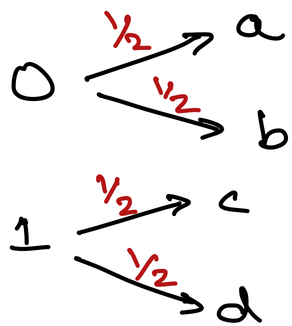


$$\max_{p(x)} I(X;Y)$$

$$= \max_{p(x)} H(X)$$

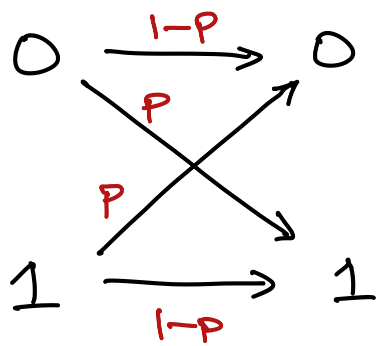
$$= 1$$

Noiseless, random



$$= 1$$

Binary Symmetric

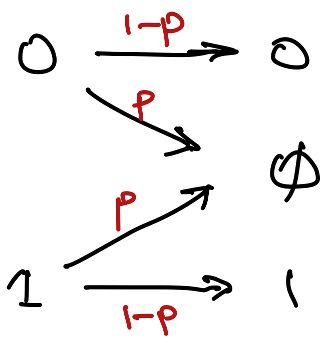


$$I(X;Y)$$

$$= H(X) - H_2(p)$$

$$C = 1 - H_2(p)$$

Binary Erasure

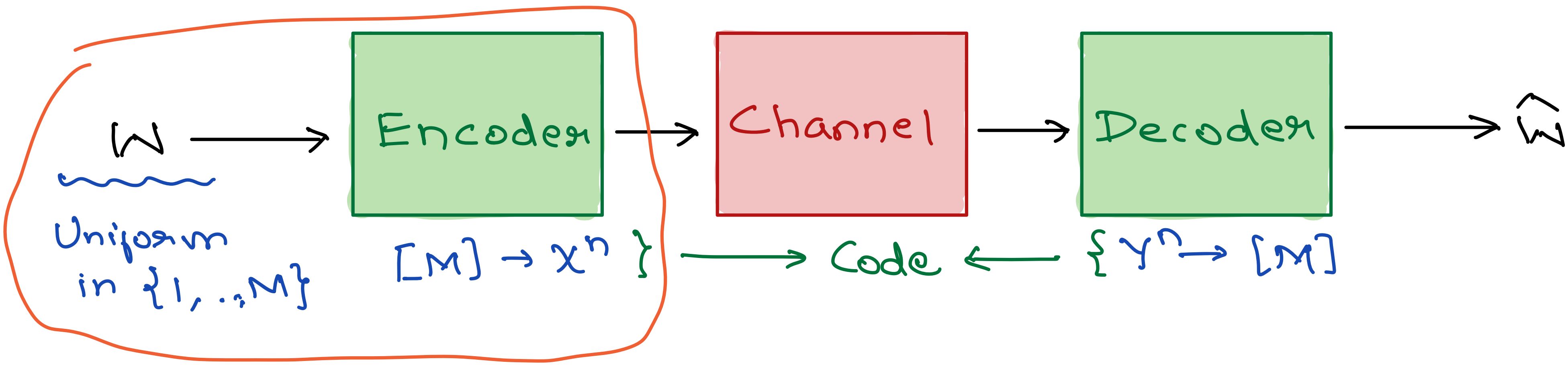


$$H(X;Y) = p \cdot H(X)$$

$$I(X;Y) = (1-p)\,H(X)$$

$$C = 1 - p$$

Channel Coding Theorem



$\log|M| = n \cdot R$

$M = 2^{n \cdot R}$

- $P_e = \mathbb{P}[W \neq \hat{W}]$

- Achievable rate R: $\exists$ Sequence of codes, rate $\geq R$

  $P_e^{(n)} \to 0$ as $n \to \infty$

▸ $\sup\{R \mid R \text{ is achievable}\} = R^* = C = \max_{P(x)} I(X;Y)$

Achievable $R \le C$

$$\boxed{W \longrightarrow \bar{X}} \longrightarrow \bar{Y} \longrightarrow \hat{W} \qquad P_e = \mathbb{P}[W \neq \hat{W}]$$

Fano:

$$\underbrace{H_2(P_e)}_{1 \ge} + P_e \cdot \underbrace{\log M}_{nR} \ge H(W \mid \hat{W})$$

$$= \underbrace{H(W)}_{nR} - I(W; \hat{W})$$

$$\ge nR - I(\bar{X}; \bar{Y})$$

$$I(\bar{X};\bar{Y}) = H(\bar{Y}) - H(\bar{Y}\mid\bar{X})$$

$$= \sum_i \underbrace{H(Y_i \mid Y_1 \ldots Y_{i-1})}_{\leq H(Y_i)} - \sum_i \underbrace{H(Y_i \mid Y_1 \ldots Y_{i-1}, \bar{X})}_{= H(Y_i \mid X_i)}$$

$$\leq \sum_i I(X_i; Y_i)$$

$$\leq n \cdot C$$

$$1 + P_e \cdot nR \geq nR - n \cdot C$$

$$\underbrace{\frac{1}{n}}_{\to 0} + \underbrace{P_e}_{\to 0} \cdot R + C \geq R$$