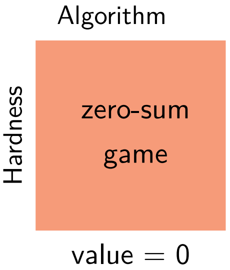
A Characterization of Strong Approximation Resistance



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TTI Chicago

Joint work with
Subhash Khot and Pratik Worah

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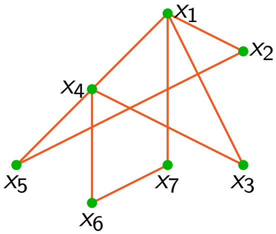

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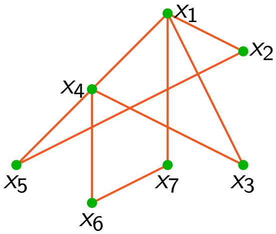

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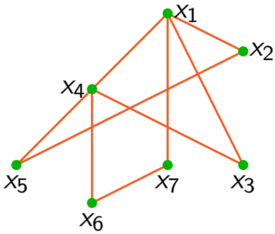

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One of the most fundamental classes of optimization problems.

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$$C_i \equiv f\left(x_{i_1} \cdot b_1^{(i)}, \ldots, x_{i_k} \cdot b_k^{(i)}\right)$$

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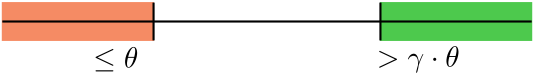

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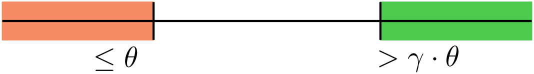

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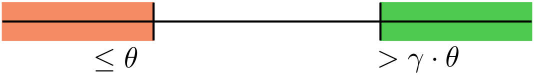

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- Can solve for all $\theta \implies$ Can approximate within factor γ.
- Hard to solve for some $\theta \implies$ Hard to approximate within factor γ.

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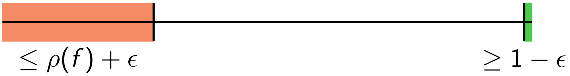

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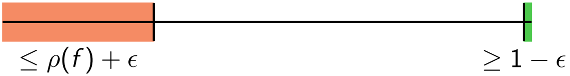

- Captures the notion of when is it hard to do better than a random assignment.

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- [AK 13*]: Characterization when f is even and instance is required to be k-partite.

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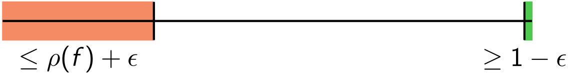

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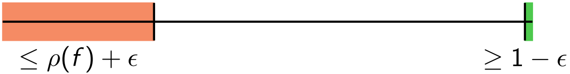

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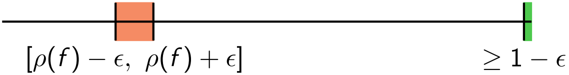

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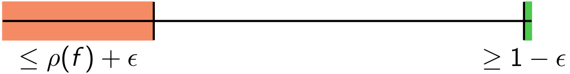

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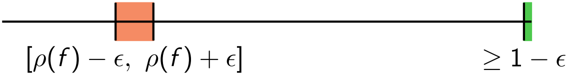

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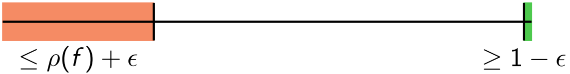

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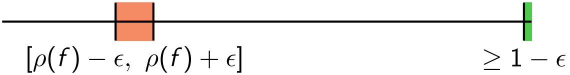

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- Equivalent to approximation resistance for odd predicates. Almost all previous results prove strong approximation resistance.

A partial characterization by [Rag 08] and [RS 09]

- [Rag 08*]: f is approximation resistant iff $\forall \epsilon > 0$ there exists a $1 - \epsilon$ vs. $\rho(f) + \epsilon$ integrality gap instance for a certain SDP.

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- Is it just properties of f or is the topology of the instance also important? (Hint: Just f)

The Austrin-Mossel condition in a new language

- For a distribution μ on $\{-1,1\}^k$, let $\zeta(\mu) \in \mathbb{R}^{k+\binom{k}{2}}$ denote the vector of first and second moments

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- Our condition is in terms of existence of a measure Λ on $\mathcal{C}(f)$ with certain symmetry properties.

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- If Λ is supported only on 0, then so is each $\Lambda_{S,\pi,b}$. If Λ is supported only on (say) (1,. . . ,1) then $\Lambda_{[k],\mathrm{id},b}$ is supported only on the point $(b_1, \ldots, b_k, b_1 \cdot b_2, \ldots, b_{k-1} \cdot b_k)$

Our Characterization

- Recall that $f : \{-1, 1\}^k \to \{0, 1\}$ can be written as

$$f(x) = \sum_{S \subseteq [k]} \widehat{f}(S) \cdot \prod_{i \in S} x_i = \rho(f) + \sum_{t=1}^{k} \sum_{|S|=t} \widehat{f}(S) \cdot \prod_{i \in S} x_i$$

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- [KTW 13*]: f is strongly approximation resistant if and only if there exists a measure Λ on $\mathcal{C}(f)$ such that for all $t = 1, \dots, k$

$$\sum_{|S|=t} \sum_{\pi : S \to S} \sum_{b \in \{-1,1\}^S} \widehat{f}(S) \cdot \left(\prod_{i \in S} b_i \right) \cdot \Lambda_{S,\pi,b} \equiv 0$$

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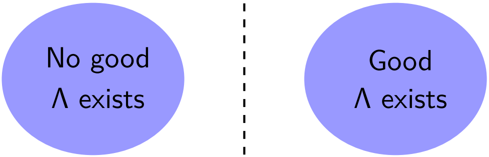
- If $|S| = t$, then $\Lambda_{S,\pi,b}$ is a measure on $\mathbb{R}^{t+\binom{t}{2}}$. For each t, above expression is a linear combination of such measures.

Proof Structure



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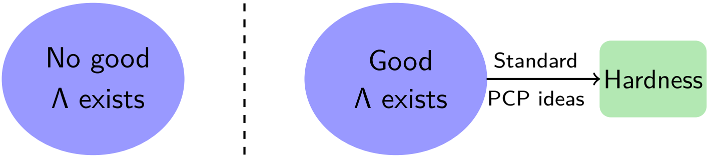

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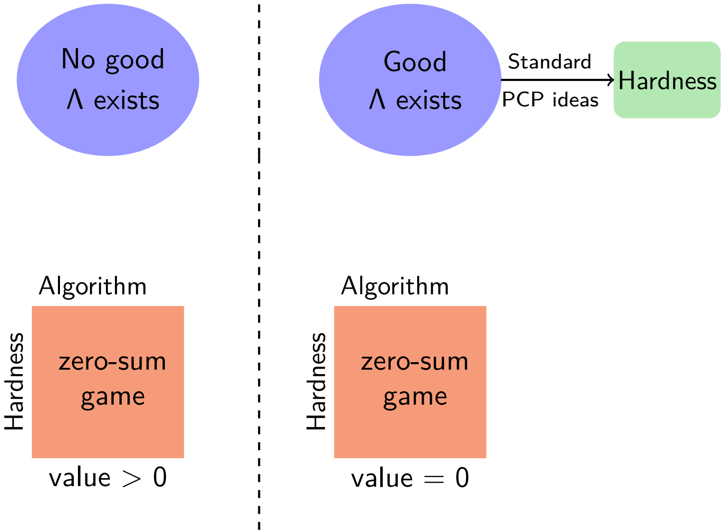

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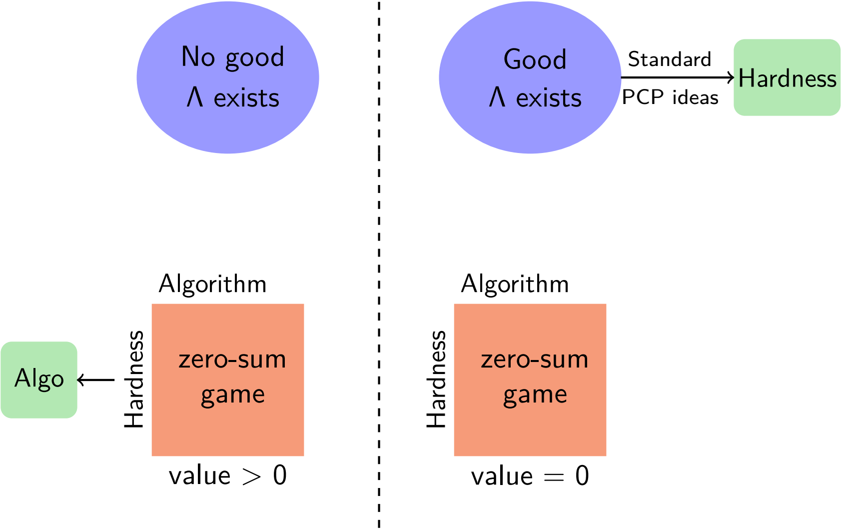

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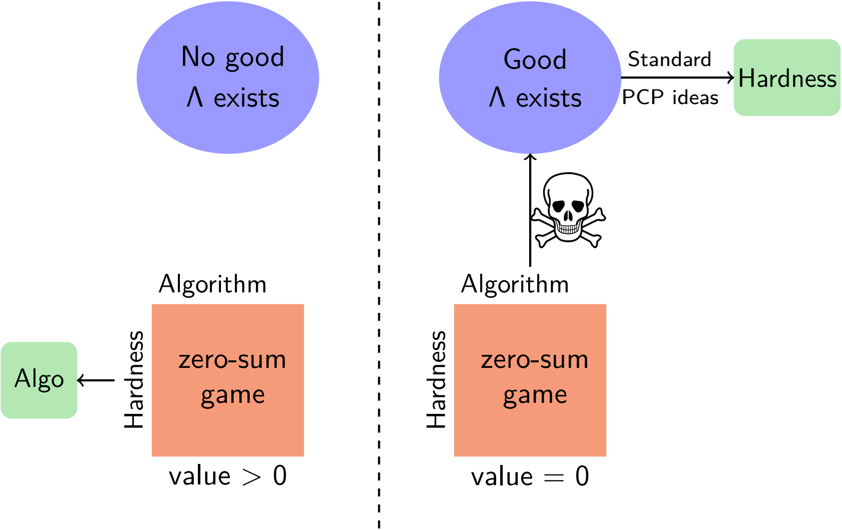

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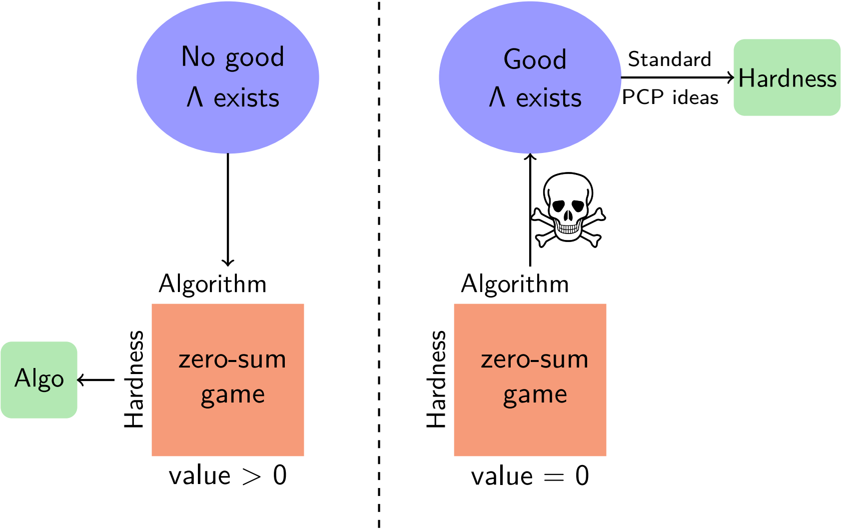

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- Value > 0 implies (a distribution over) rounding strategies which show that predicate is not strongly approximation resistant. (since every instance corresponds to a Λ)

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- Bulk of the work in analyzing sequence of finite games and coefficients of corresponding polynomials.

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Thank You

Questions?