Time space tradeoffs for attacks against one-way functions and PRGs

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  - Recover a key of length k in time less than 2^k.
  - In time t, recover key with probability better than $t/2^k$.
- Are better (non-uniform) attacks possible against:
  - one-way functions?
  - pseudo-random generators?

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- PRG: $G : [N] \to [2N]$ is a (t, ϵ)-secure PRG if for every algorithm A of complexity $\leq t$

$$\left| \Pr_{x \sim [N]} \left[A(G(x)) = 1 \right] - \Pr_{y \sim [2N]} [A(y) = 1] \right| \leq \epsilon$$

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- Can be implemented on a RAM with time and space t.
- Similar to circuit complexity.

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All above results are actually stated as time-space tradeoffs. Complexity is optimized when $T = S$.

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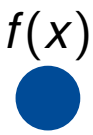

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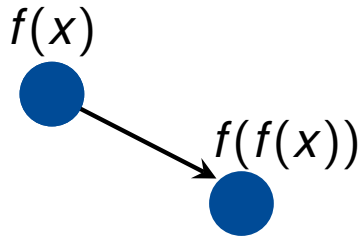

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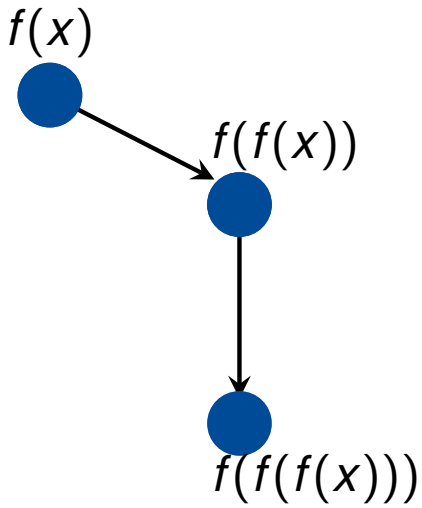

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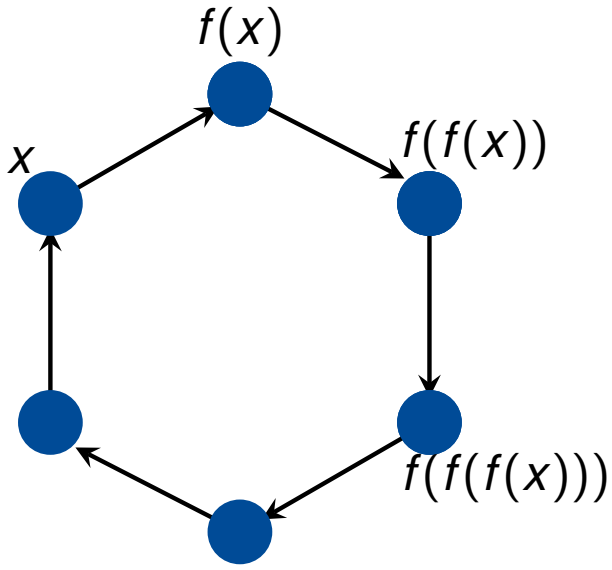

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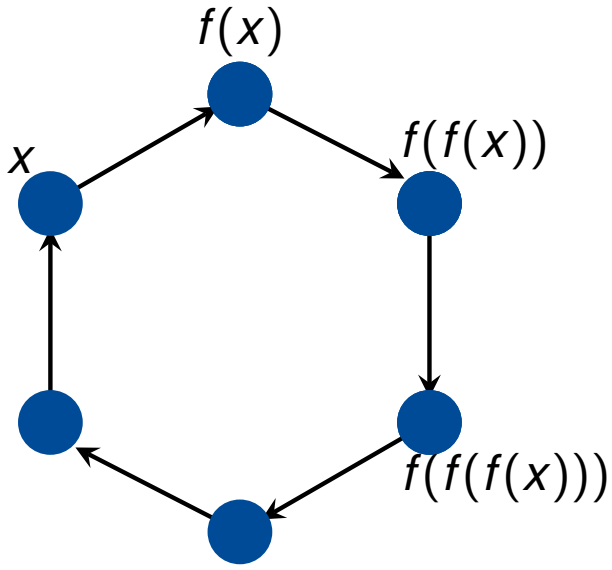

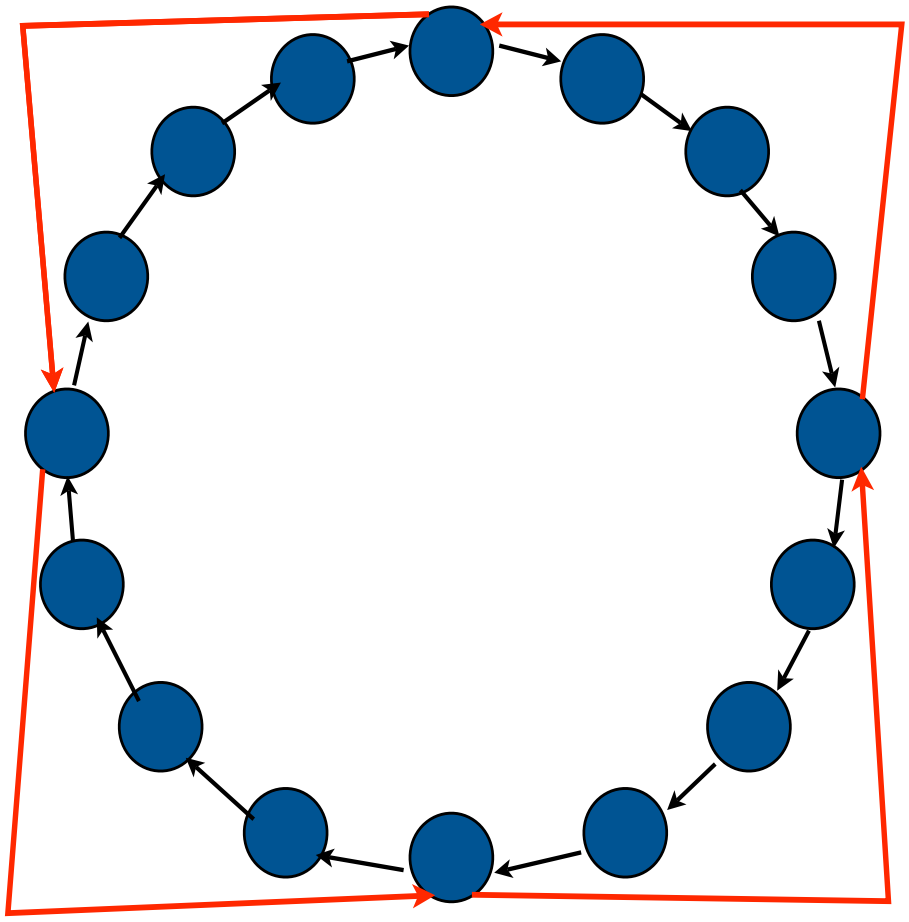

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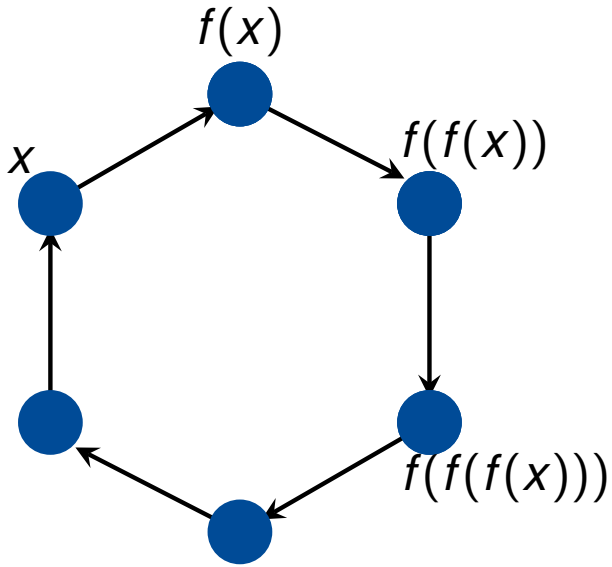

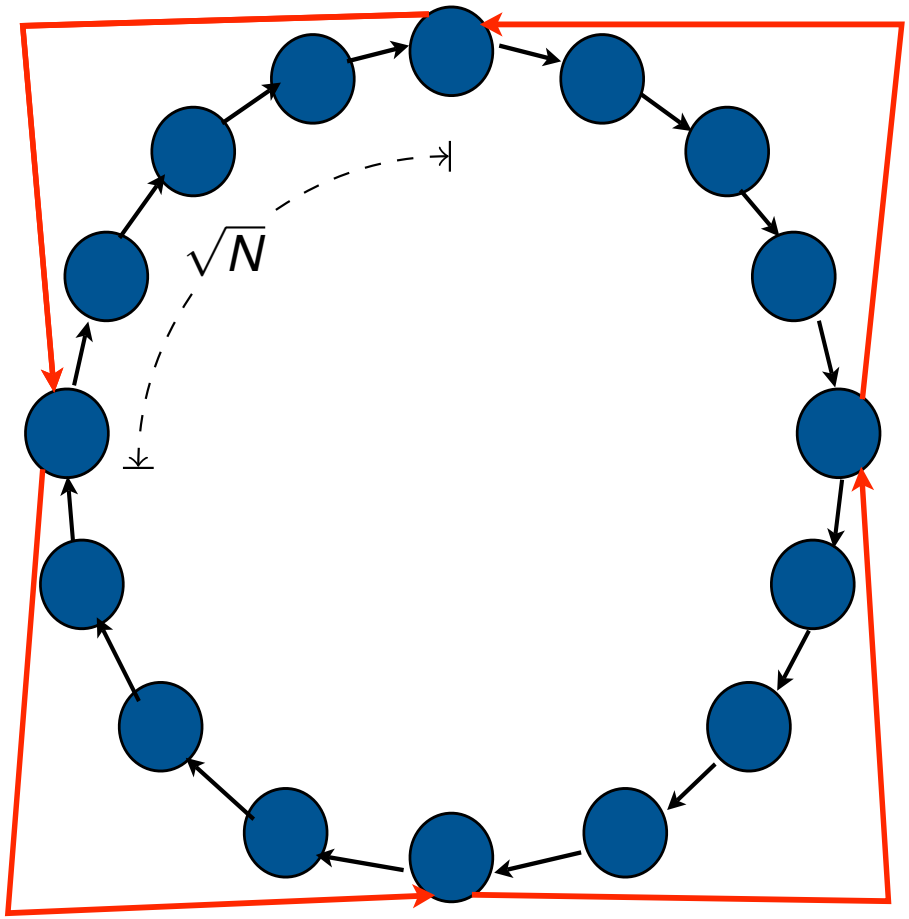

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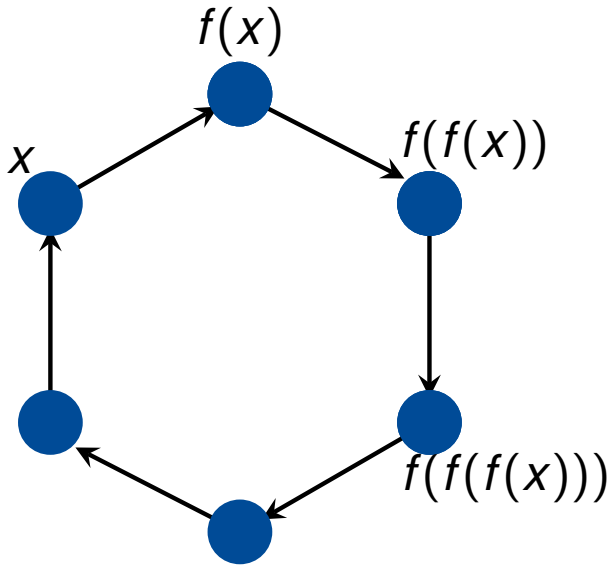

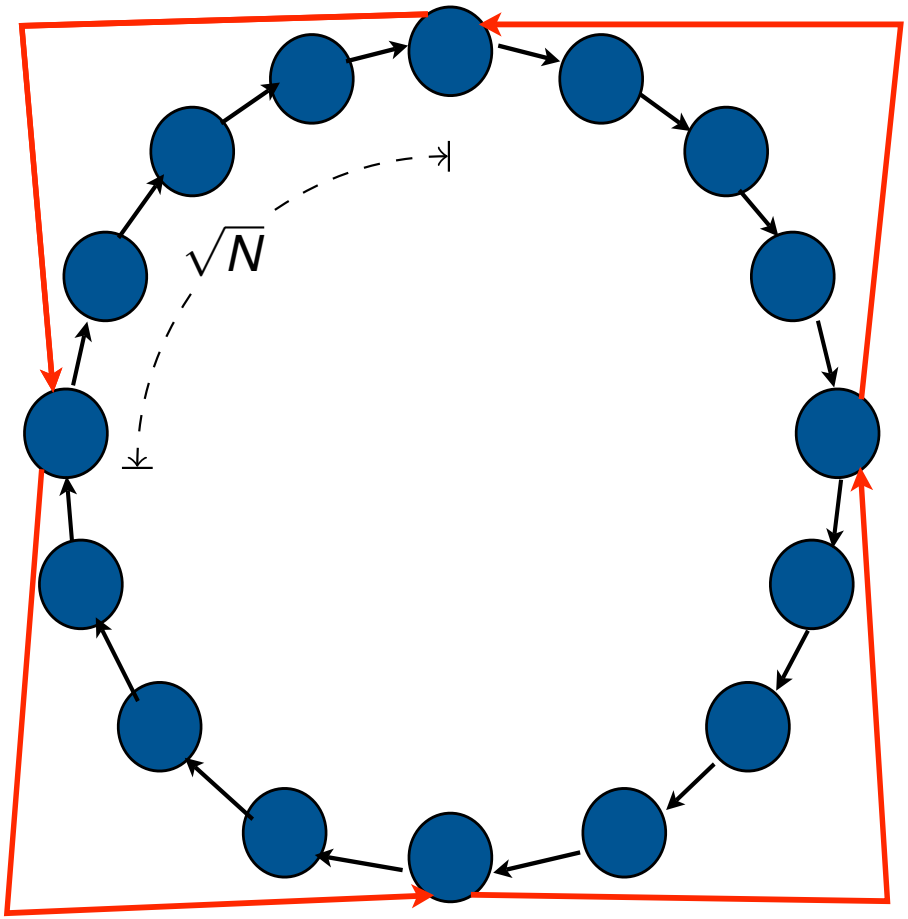

In large cycles, store back-links at distance $\sqrt{N}$. $T, S = O(\sqrt{N})$.

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- Cover the graph $(x \to f(x))$ of f by m disjoint paths of length ℓ.

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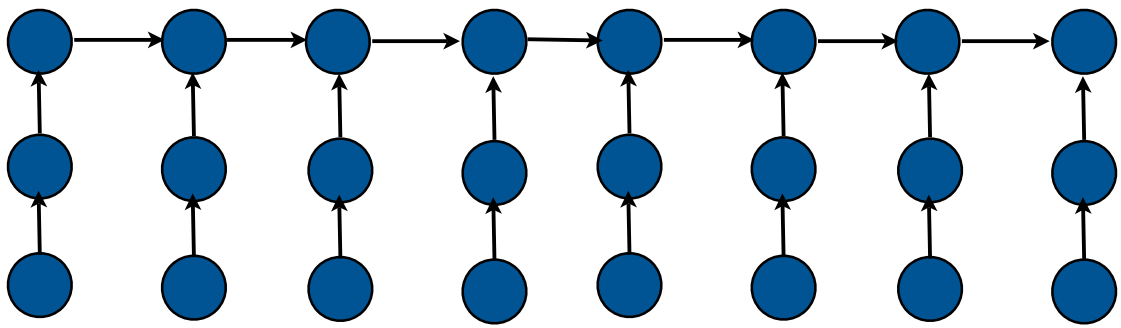
- Problem: m may have to be very large.



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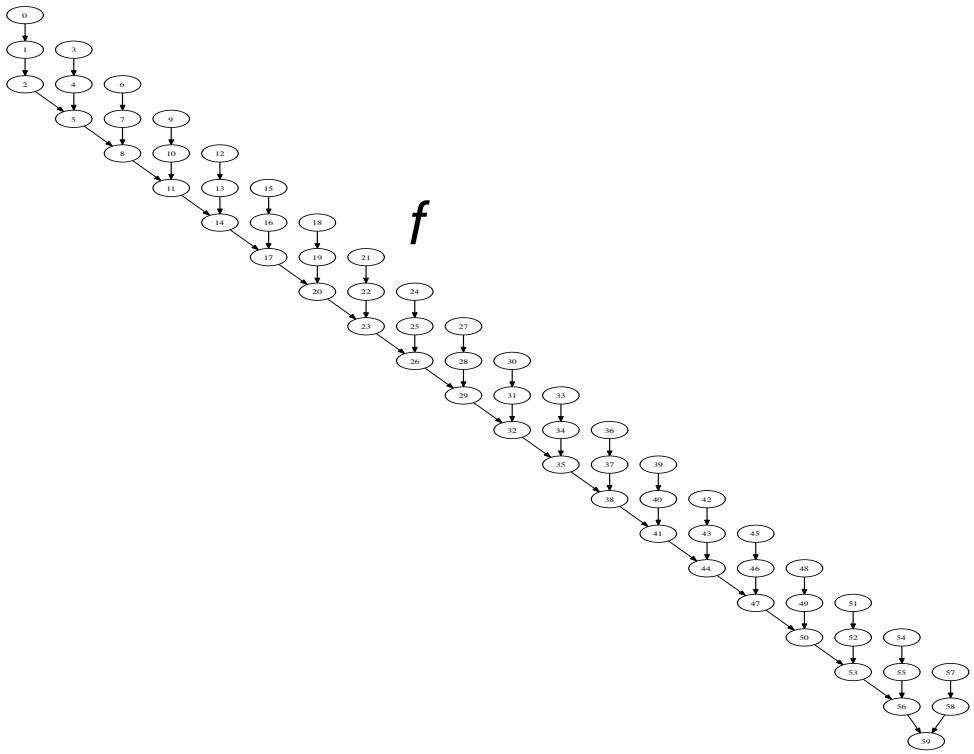

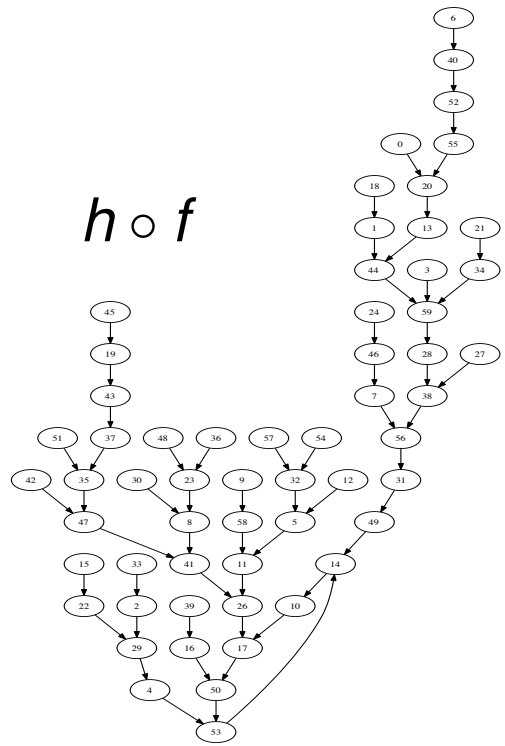

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- Choose $m, \ell, r = \tilde{O}(N^{1/3})$.

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- Problems: Computing $h_1, \ldots, h_r$. Restricted to random f.

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- Can again choose m, l such that $m\ell^2\lambda \approx m\ell^2/K \ll 1$. Can get

$$T, S = \tilde{O}(N^{3/4})$$

by taking $K = \tilde{O}(N^{3/4})$, $r = \tilde{O}(N^{1/2})$ and $m, \ell = \tilde{O}(N^{1/4})$.

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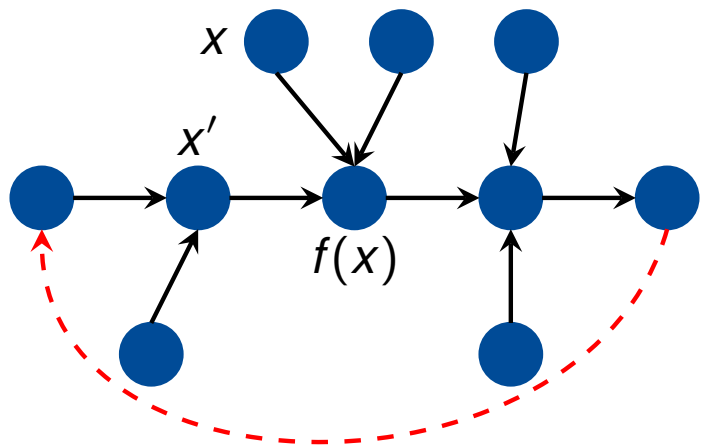
- First observation: If a table of size K does not invert f with probability ϵ, then the collision probability for the rest is ϵ/K.
- Second Observation: The number of elements inverted by a path is not just the path length, but the the sum of indegrees of elements in the path.



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- Final complexity: $T, S = \begin{array}{ll} \tilde{O}(\sqrt{\epsilon N}) & \epsilon \leq N^{-1/3} \\ \tilde{O}(\epsilon^{5/4} N^{3/4}) & \epsilon \geq N^{-1/3} \end{array}$

Lower bound for inverting permutations

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- Give a simpler, "randomized" proof that works for all T. Also extends to lower bounds for PRGs.
- As in [GT00], show that using A, can encode f with $\approx \log(N!) - \frac{\epsilon N}{100T} + S$ bits. Thus, $S > \frac{\epsilon N}{100T}$.

Intuition for the encoding

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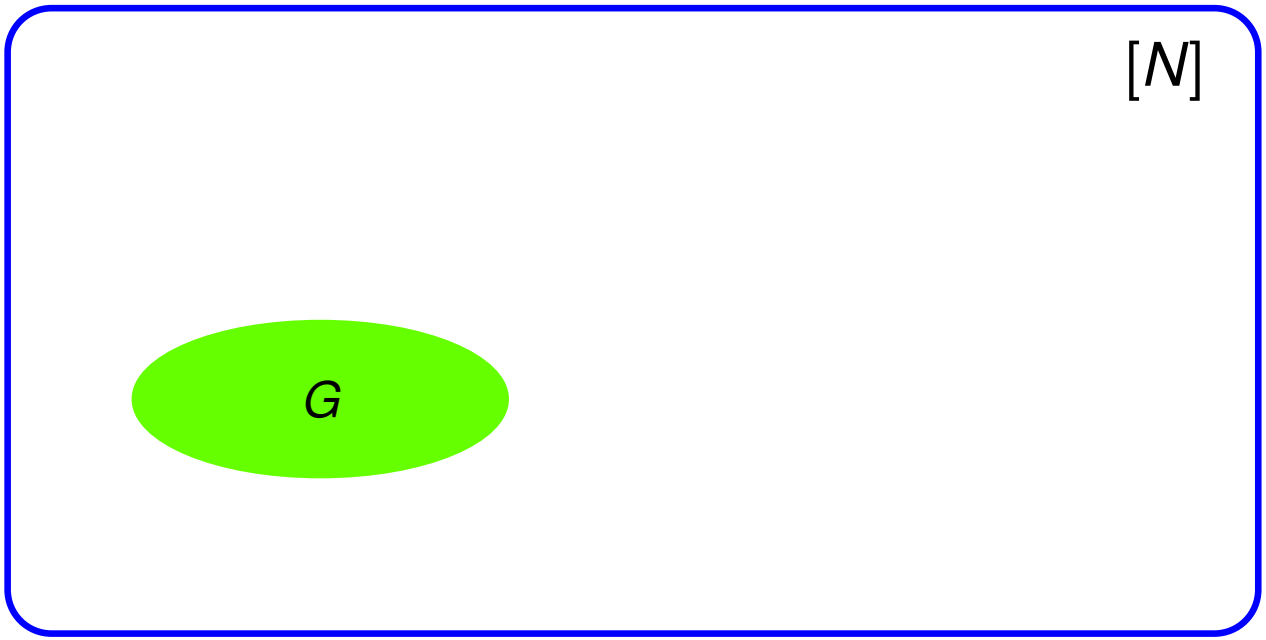

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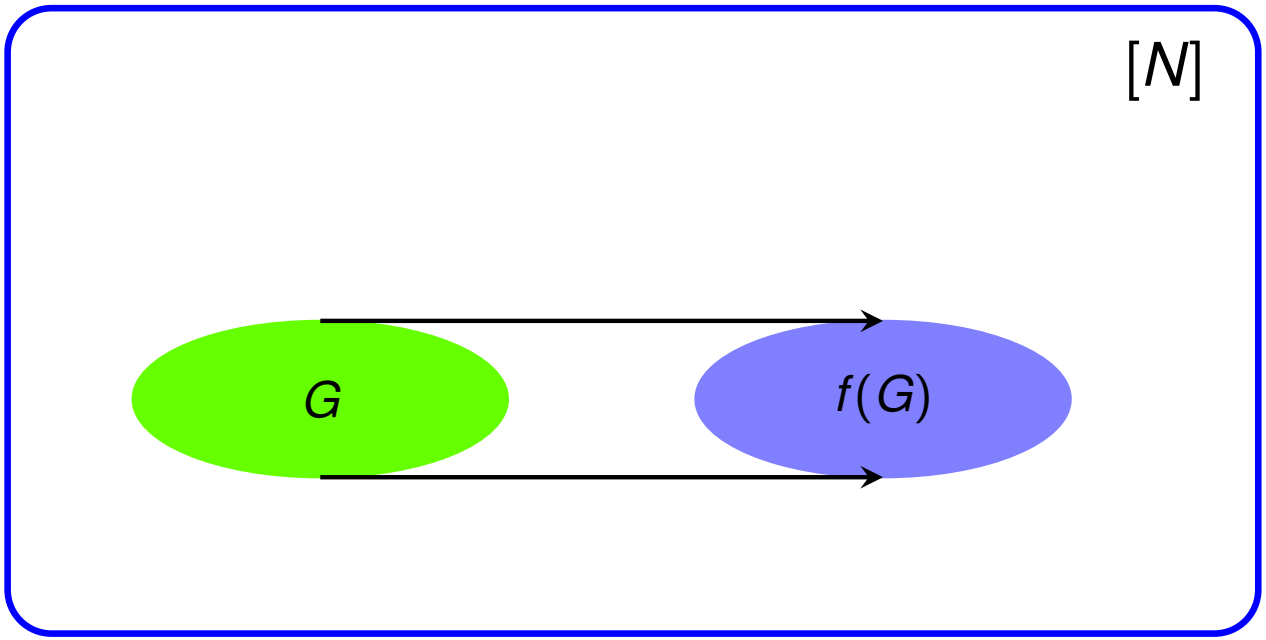

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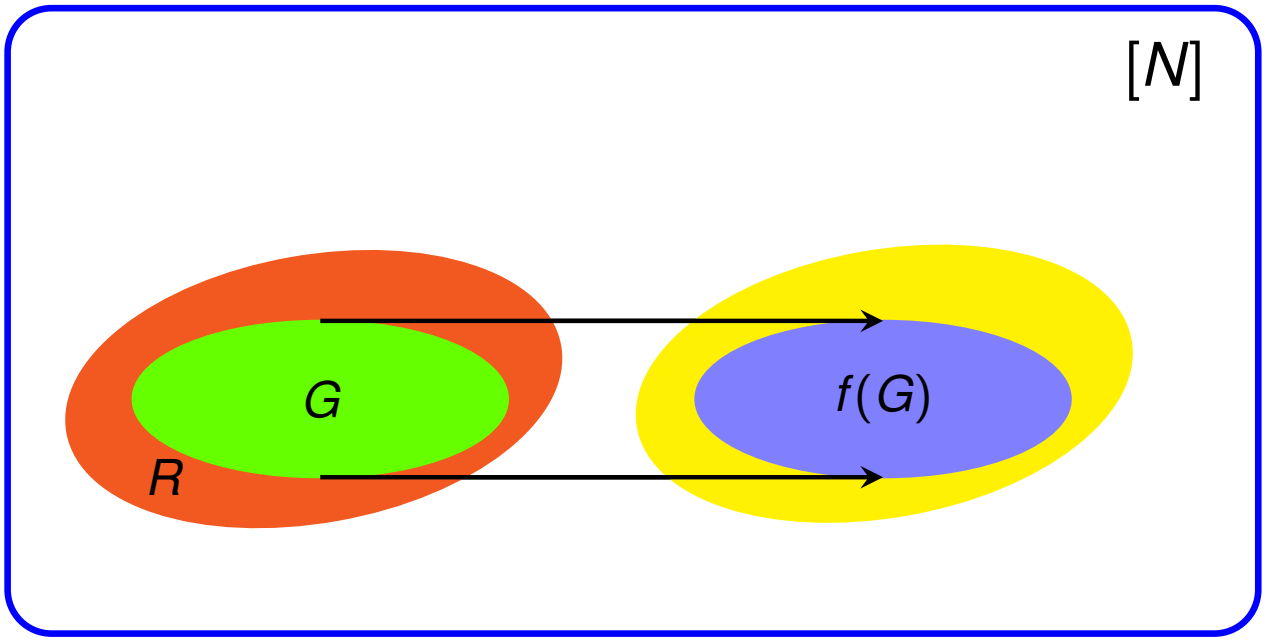

$$|G| = \frac{\epsilon N}{100T}$$

$$|R| = \frac{N}{10T}$$

- A inverts G correctly.
- Set R is chosen at random. $G \subseteq R$.
- For all $x \in G$, A does not query any element in R.

Conclusions

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- The best provable upper bound for one-way functions on all inputs remains $N^{3/4}$ and $N^{2/3}$ is the best for "Hellman"-style arguments (Barkan, Biham and Shamir)
- Techniques for proving lower bounds do not seem to do any better for one-way functions than permutations i.e. $\Omega(N^{1/2})$.

Thank You

Questions?