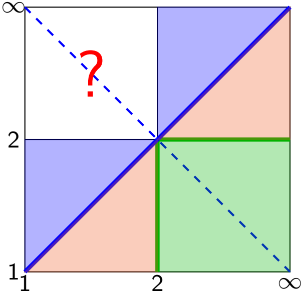
(In)Approximability of Matrix Norms



Madhur Tulsiani

Joint work with

Vijay Bhattiprolu, Mrinalkanti Ghosh,
Venkatesan Guruswami and Euiwoong Lee

Matrix Norms

- Given $A \in \mathbb{R}^{m \times n}$, find (or approximate)

$$\|A\|_{p \to q} \;:=\; \max_{x \neq 0} \frac{\|Ax\|_q}{\|x\|_p} \;=\; \max_{\|x\|_p = 1} \|Ax\|_q$$

where, $\|x\|_p \;=\; \left(\mathbb{E}_{i \in [n]} |x_i|^p\right)^{1/p}$. $\qquad (p, q \geq 1)$.

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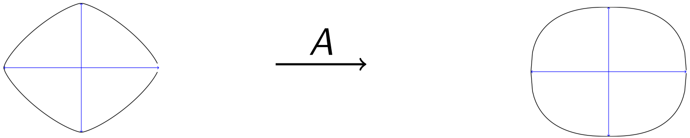
- How A maps $\|\cdot\|_p$ into $\|\cdot\|_q$.



$p = 4/3$ $\qquad$ $q = 5/2$

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- $p \geq q$: Widely studied in terms of algorithms and hardness.
- $p < q$: Hypercontractive norms. Relevant for small-set expansion like problems.

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\|A\|_{p \to q} &= \max_{\|x\|_p \le 1} \|Ax\|_q \\
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Hypercontractive remains hypercontractive ($p \le q \implies q^* \le p^*$).

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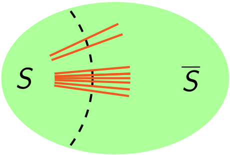

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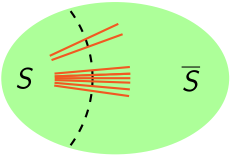

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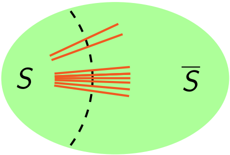

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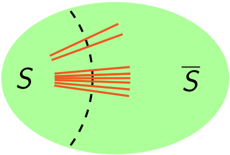

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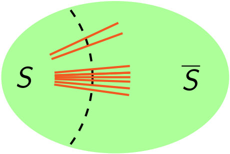

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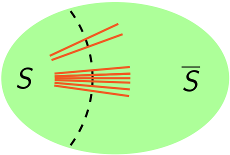

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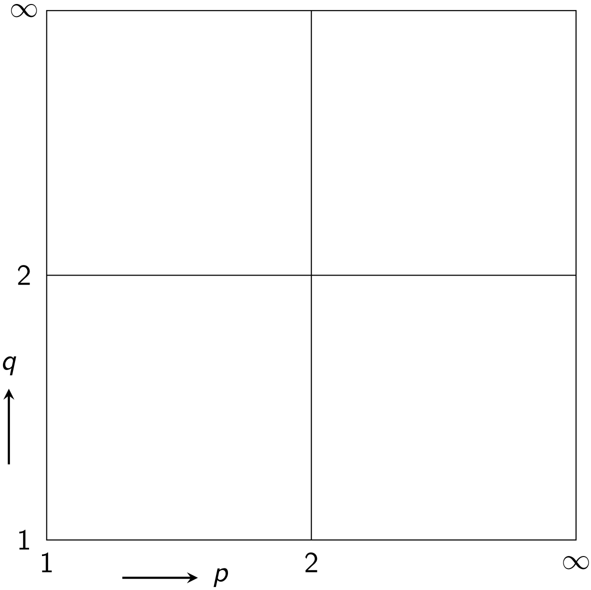
- [BBHKSZ 12]: Two-sided connection for $2 \to q$ norm of related matrix A'_G.

Known results



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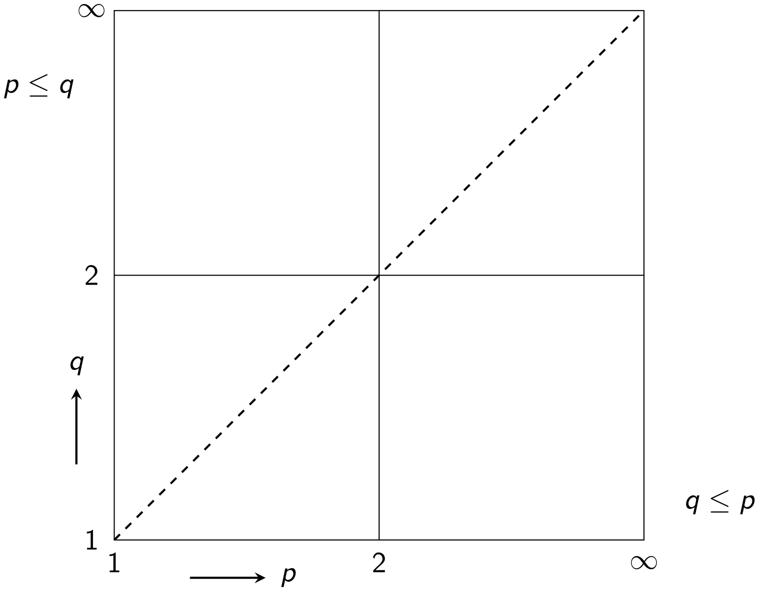

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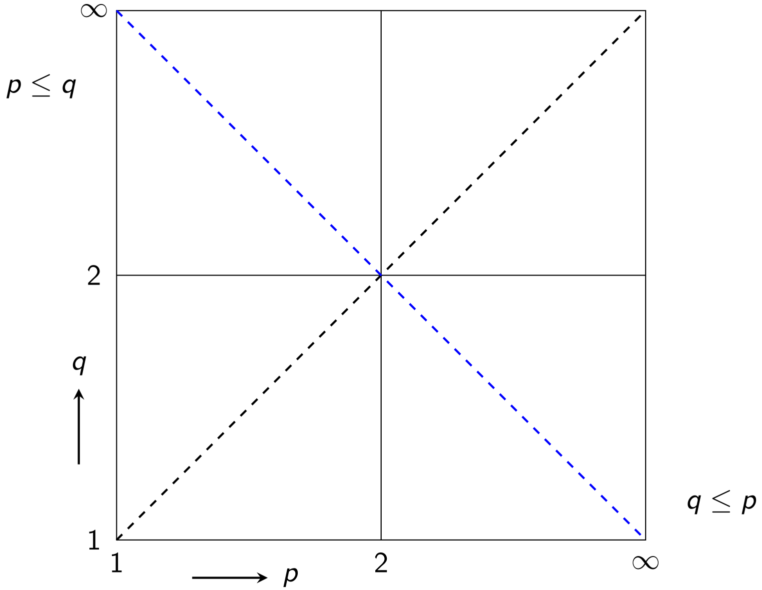

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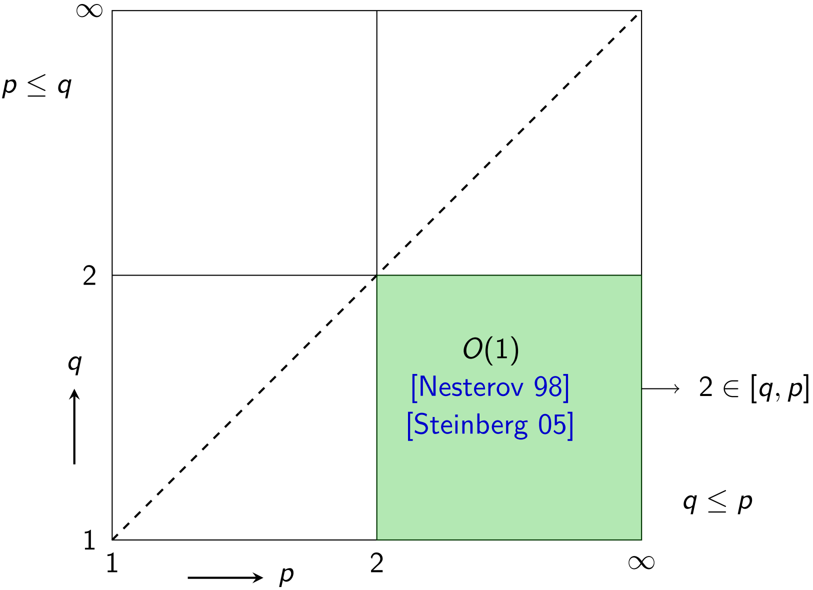

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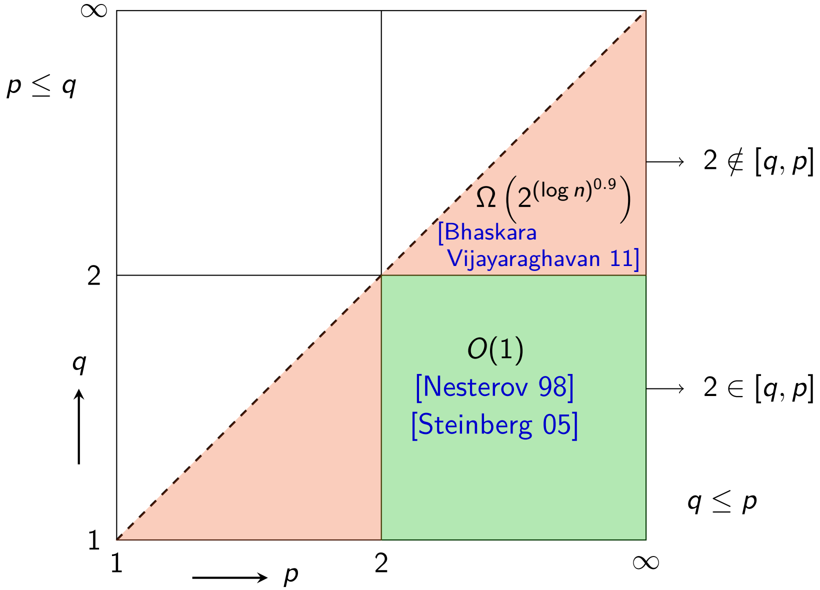

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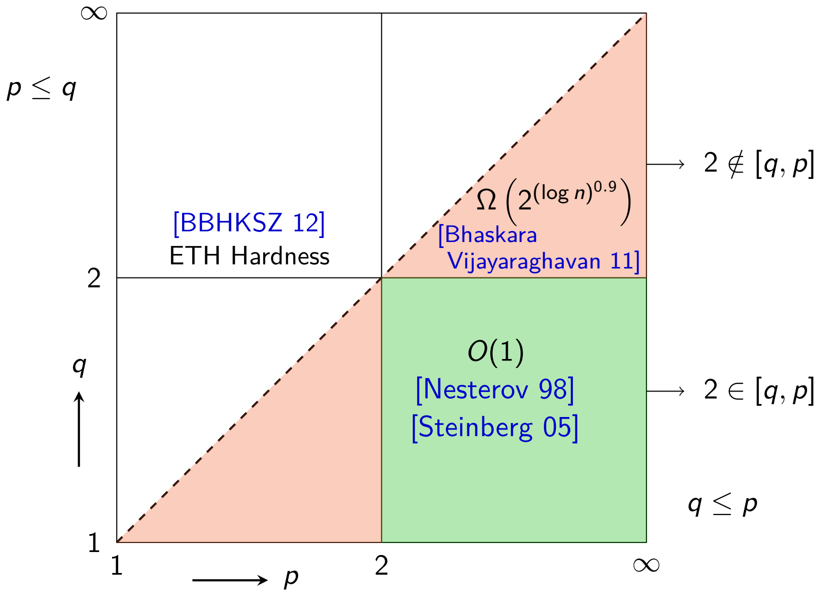

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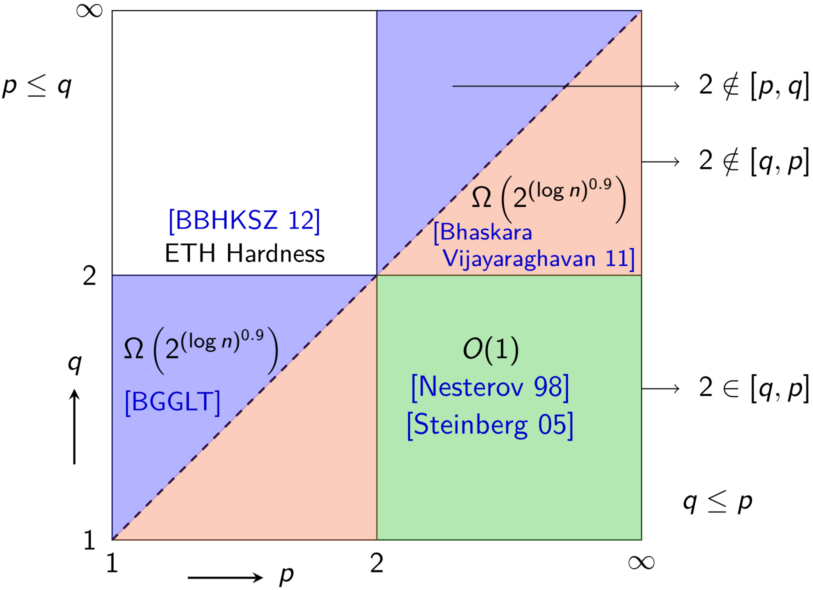

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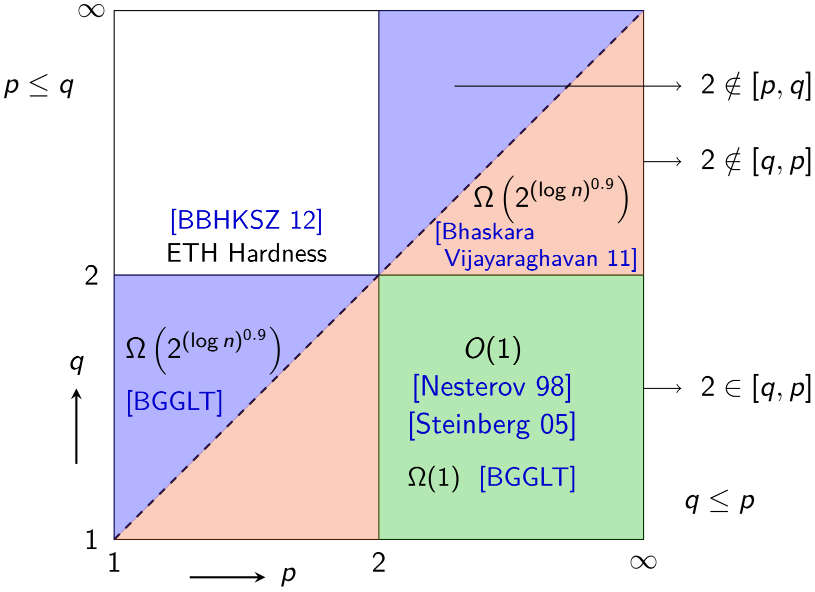

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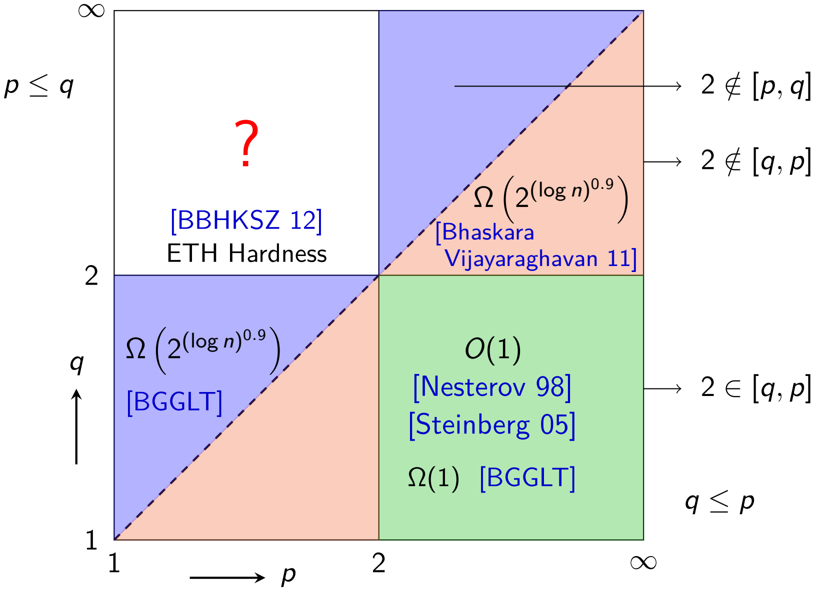

Results, implications and speculations

- $\|A\|_{p\to q}$ is hard to approximate within $2^{(\log n)^{1-\epsilon}}$ for any $\epsilon > 0$ when $2 \notin [p, q]$ (assuming NP not in randomized quasipolynomial time). Hardness of $(2+\delta) \to q$ norm for any $\delta > 0$.

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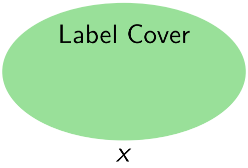
- Some evidence that the case when $2 \in [p,q]$ may be different from $2 \notin [p,q]$?

Hardness of proving hypercontractive hardness (via gadgets)



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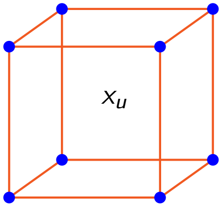

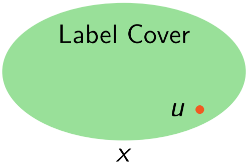

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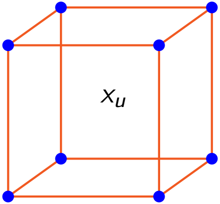

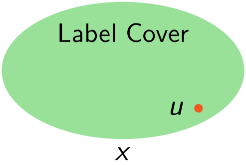

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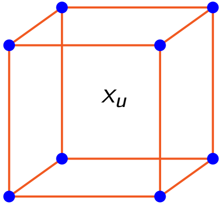

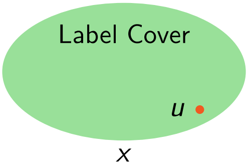

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- $\dfrac{\|x\|_q}{\|x\|_p}$ maximized when x is sparse (for $p < q$). No mass on most blocks.

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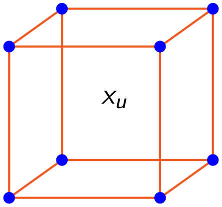

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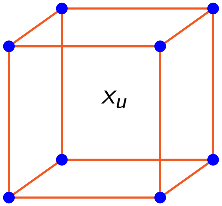

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- Rademacher gadget:

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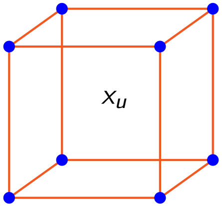

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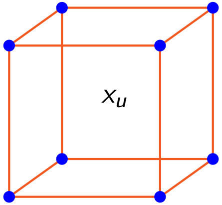

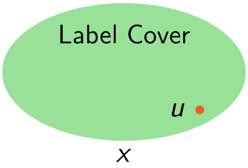

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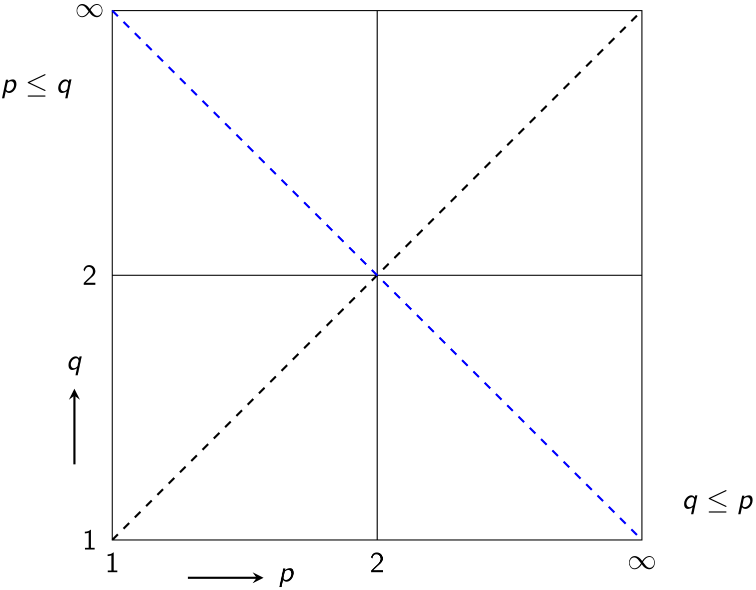
- Globally project to linear space implied by (smooth) label cover constraints. $r < 2$ ensures global spread.

Recap



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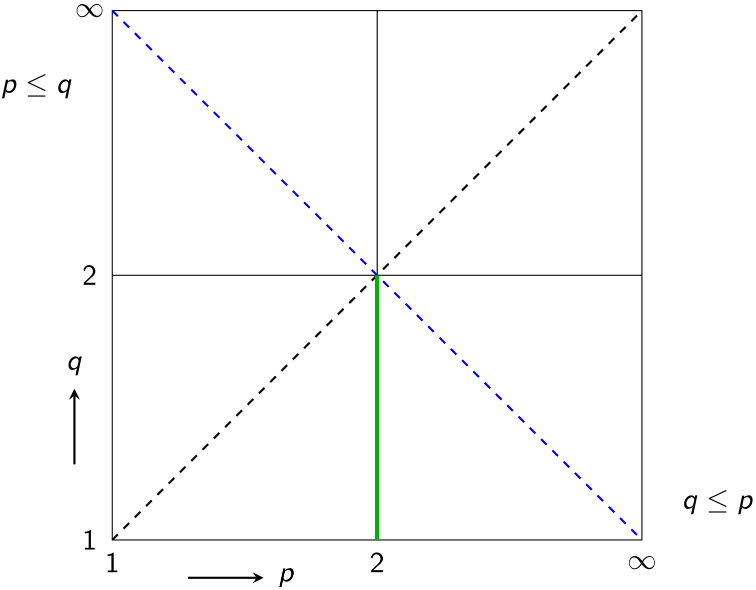

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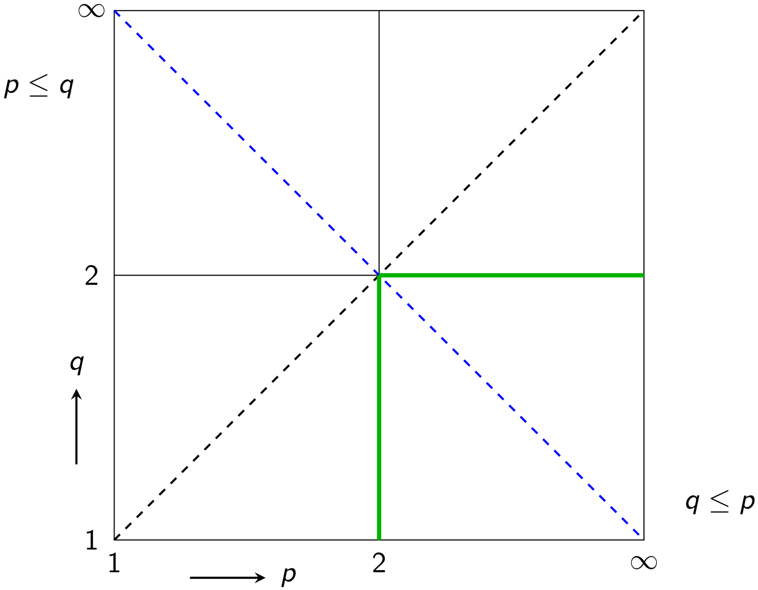

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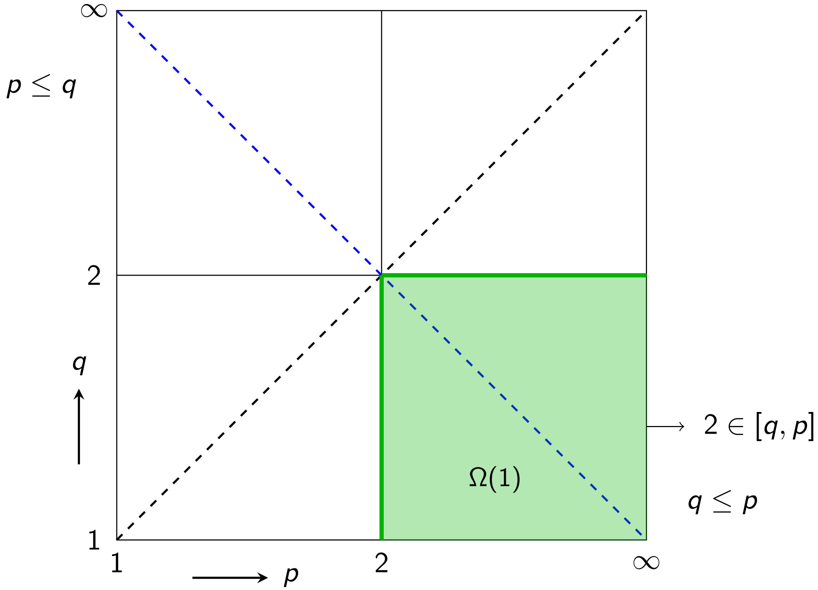

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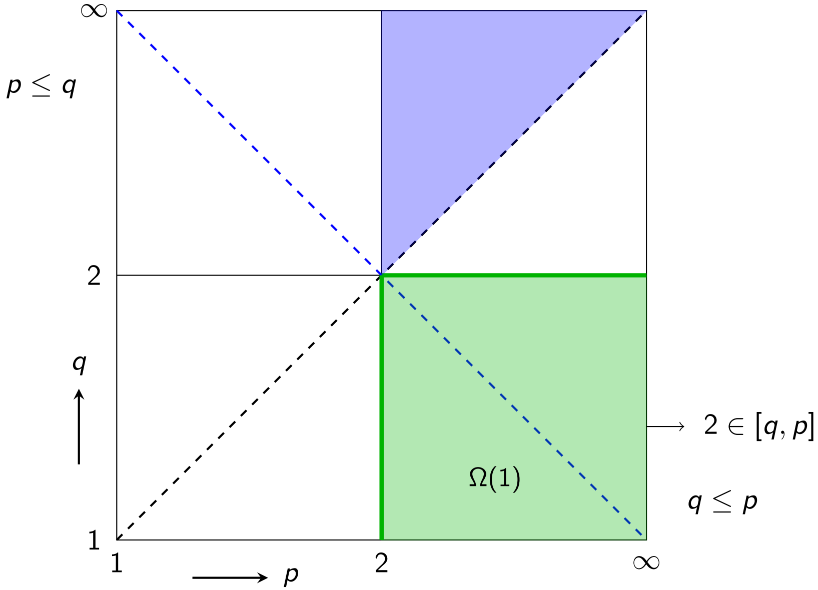

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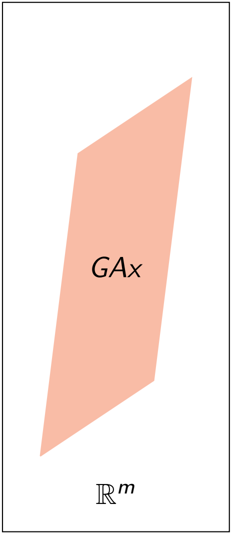

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$$\|Gz\|_q \approx \|z\|_2$$

[Dvoretzky]: Simultaneously $\forall z$ if $m \geq n^{q/2}$.

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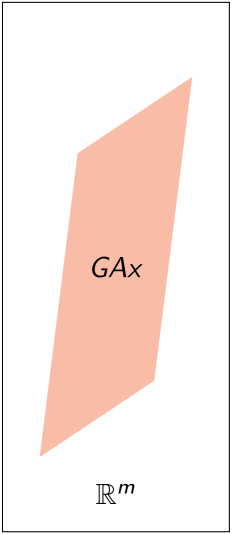

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- Reduction from $p \to 2$ norm, to $p \to q$ norm

$$\|GAx\|_q \approx \|Ax\|_2 \qquad \text{for all } x$$

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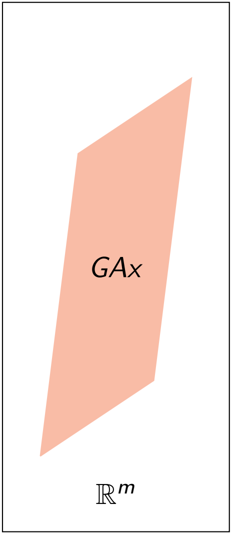
Hardness for hypercontractive norms: $2 < p \leq q$



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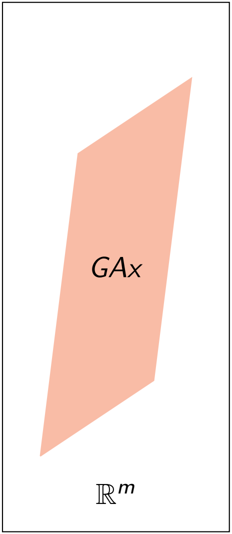

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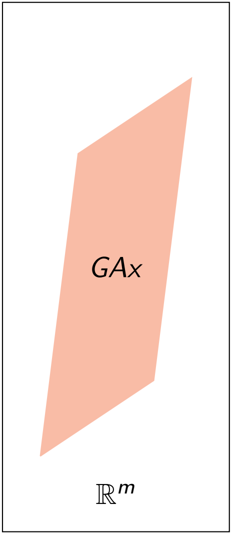

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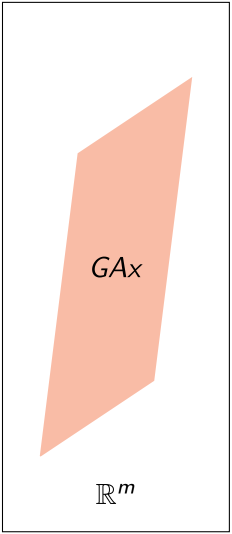

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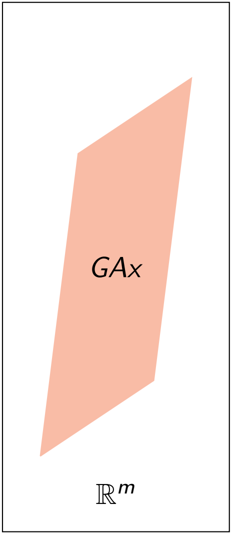

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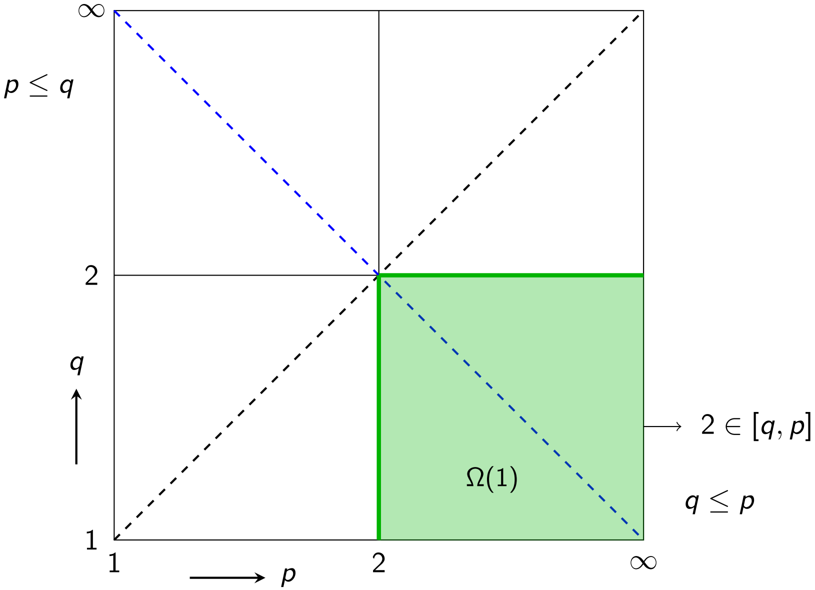
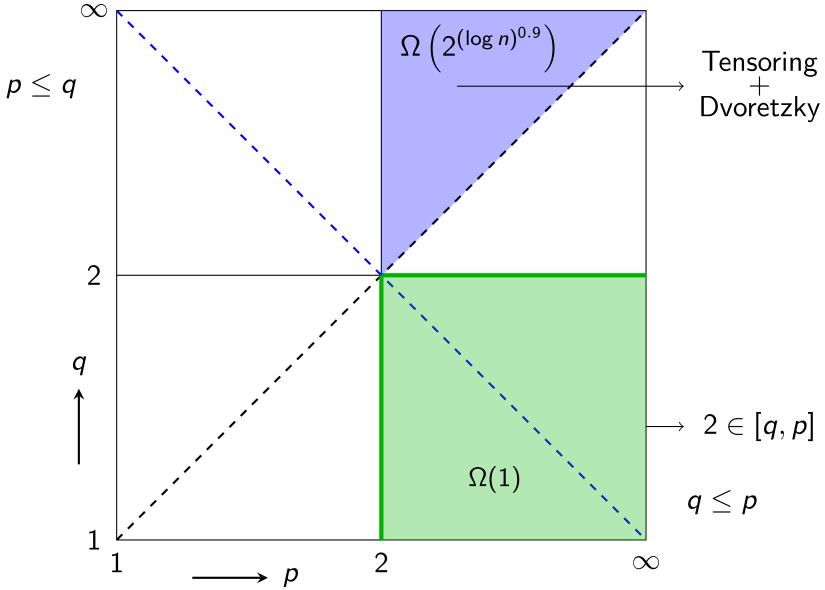
- c^t hardness for instances of size $N = n^{O(tq/2)}$. $2^{(\log N)^{1-\epsilon}}$ hardness for $t = (\log n)^{O(1/\epsilon)}$.

Recap

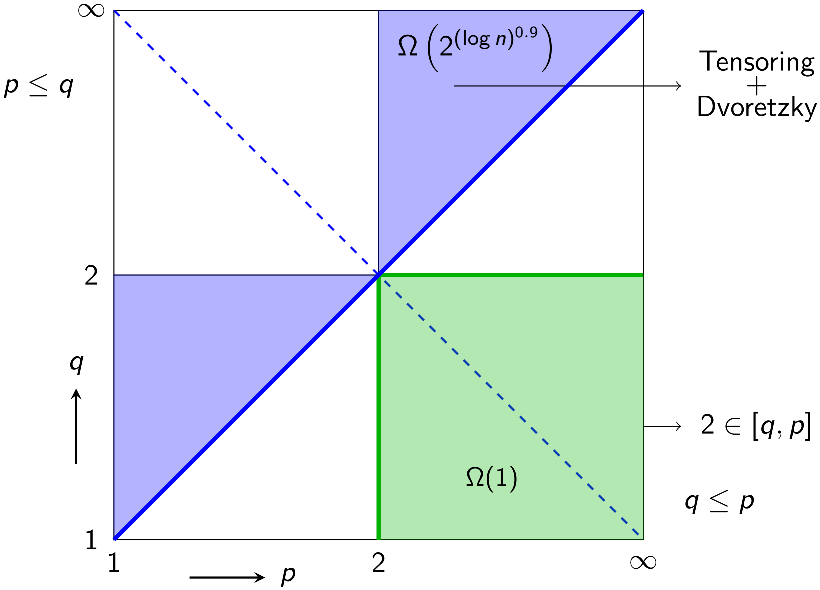


Recap



∞
$p \leq q$
2
q
1
1
p
2
∞
$\Omega\left(2^{(\log n)^{0.9}}\right)$
Tensoring
+
Dvoretzky
$2 \in [q, p]$
$\Omega(1)$
$q \leq p$

Recap



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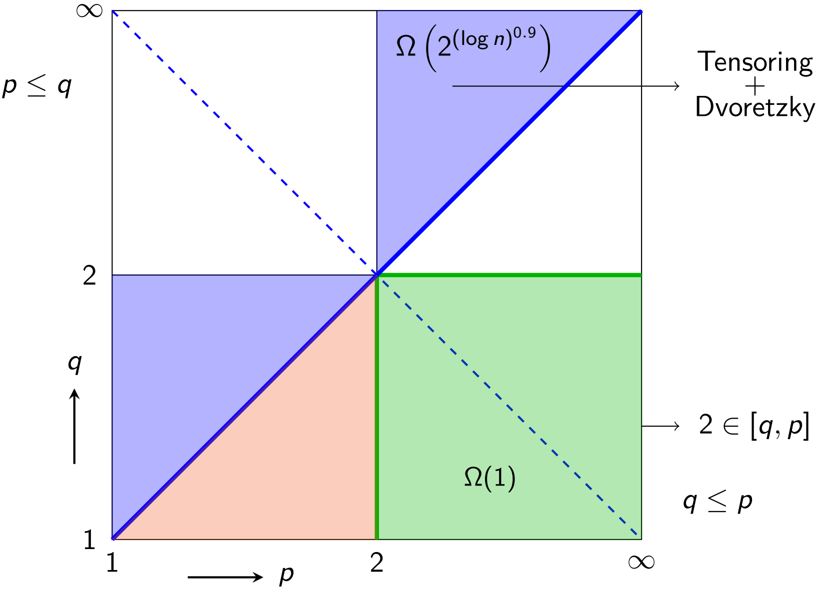

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The non-hypercontractive case: q < p < 2

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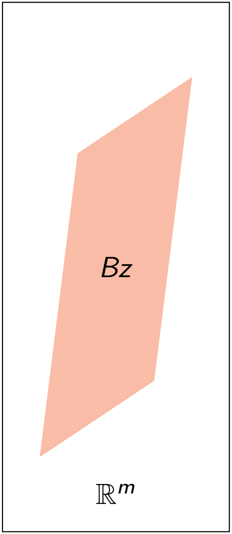

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(slight modification of Schechtman's result).

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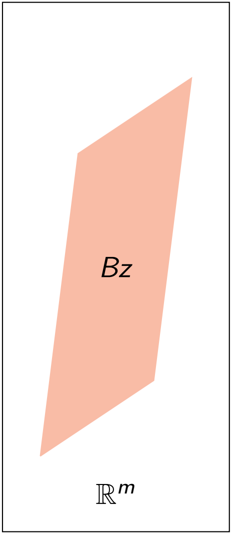

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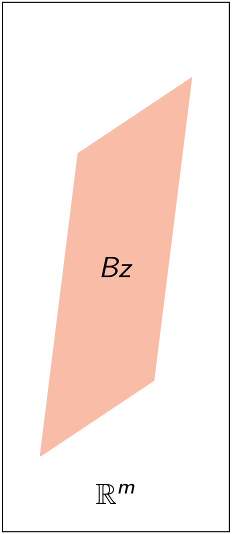

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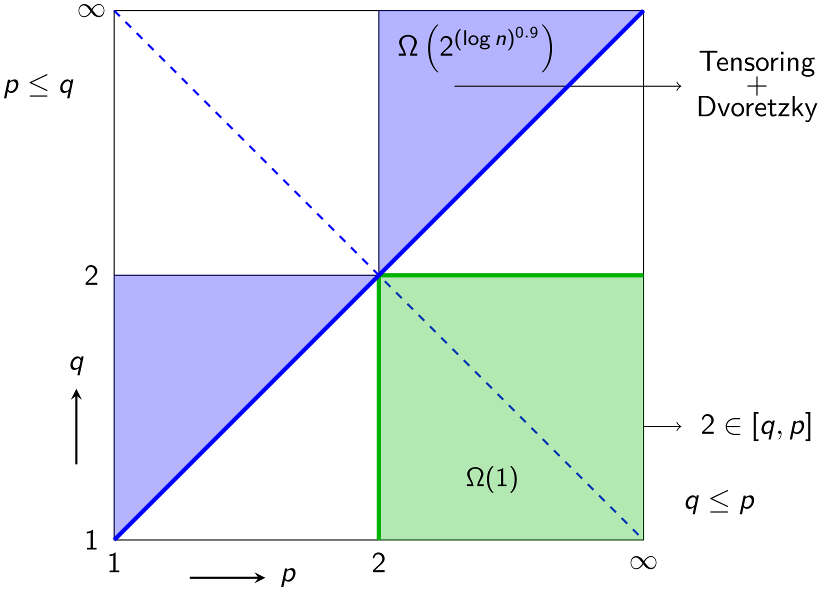
- Simplified (but randomized) proof of [BV 11].

Recap



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$\Omega\left(2^{(\log n)^{0.9}}\right)$
Tensoring
+
Dvoretzky
$p \leq q$
2
q
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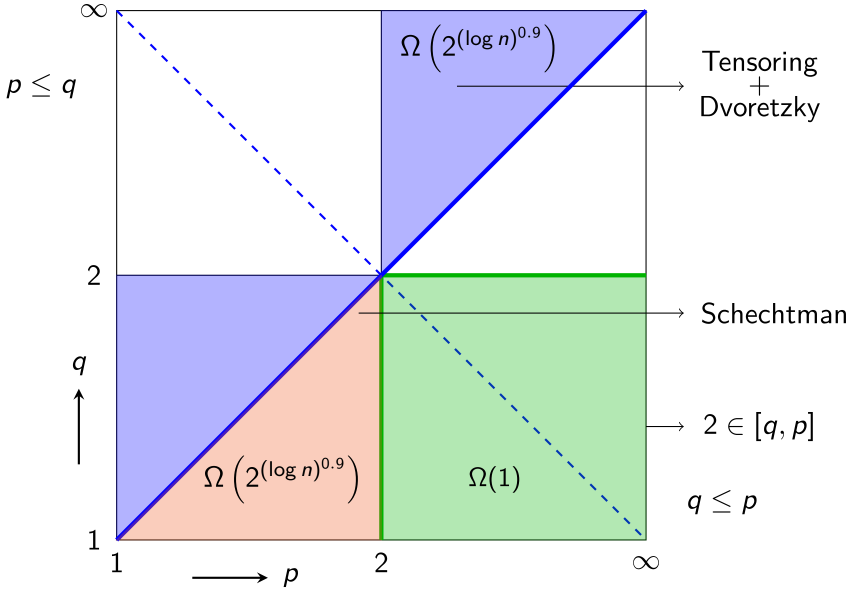
1
1
2
∞
p

Recap



∞
2
1
1
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∞
p ≤ q
q
p
Ω (2^{(log n)^{0.9}})
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Ω(1)
Tensoring
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Dvoretzky
Schechtman
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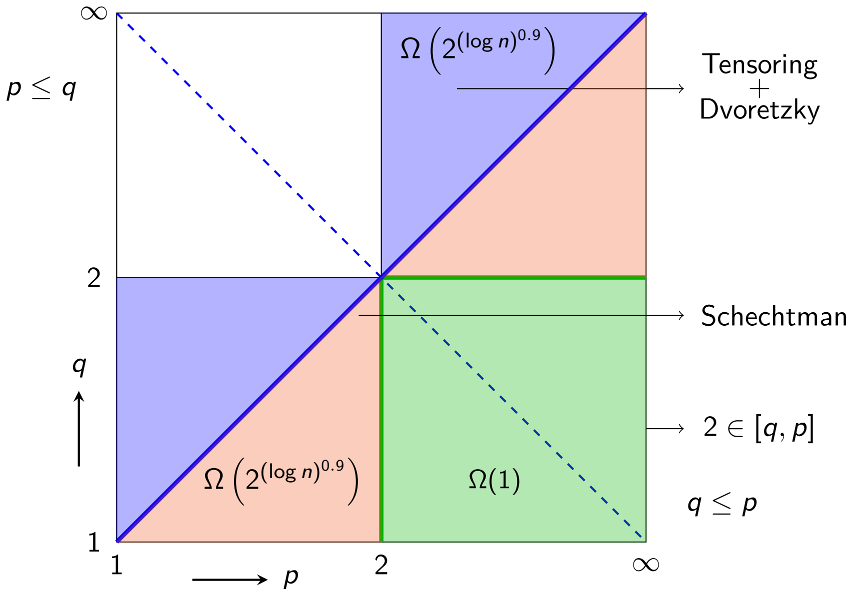
q ≤ p

Recap



∞
$p \leq q$
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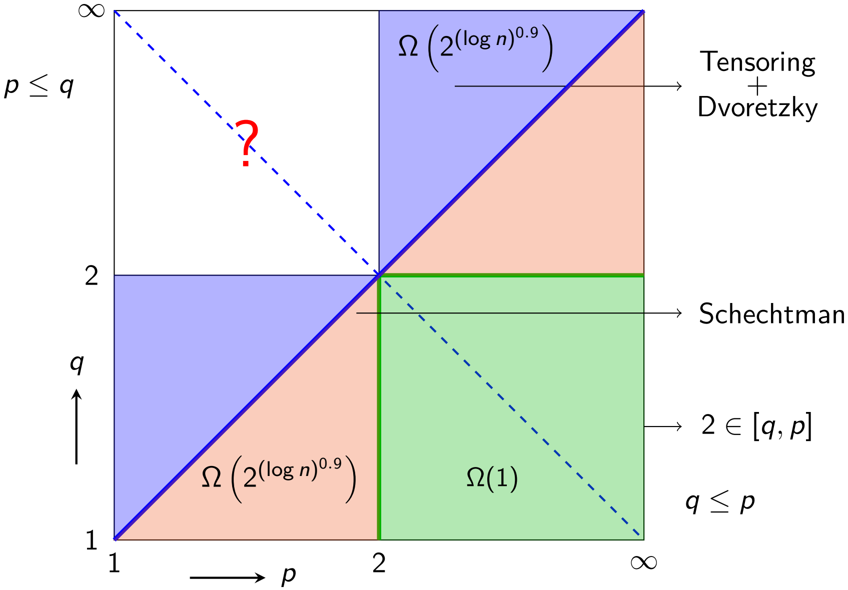
Recap



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?
Tensoring + Dvoretzky
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Open questions

- The case when $p \leq 2 \leq q$. Hardness of $2 \to 4$ norm? Of any degree-4 polynomial over ℓ_2 ball.

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- Hardness of $(2+\delta) \to 4$ norm. How small can δ be?
- Known algorithms for Unique Games, SSE imply sub-exponential algorithms for $2 \to q$ norms. Can this be extended to all $p \leq q$ when $2 \in [p, q]$?
- Right (form of) approximation ratio when $p \geq q$ and $2 \in [q, p]$.
  - Hardness result is tight when p or q equals 2.
  - [BGGLT 18]: Matching approximation ratio (up to uncertainty in value of K_G).

Thank You

Questions?