CS 39600: Theory of Metric Embeddings **Homework 1**
Date: Jan 17, 2006 **Due:** Jan 26, 2006

1. Given (X, d_1) and (Y, d_2) with $X \cap Y = \{z\}$, the *gate-sum* of the two metrics is the metric $(X \cup Y, d)$, where $d|_{X\times X} = d_1$, $d|_{Y\times Y} = d_2$, and $d(x, y) = d_1(x, z) + d_2(z, y)$ for $x \in X$ and $y \in Y$.
   (a) If both d_1 and d_2 are embeddable into ℓ_1, is d?
   (b) If both d_1 and d_2 are embeddable into ℓ_2, is d?

2. Show that Bourgain's embedding also gives
$$(X, d) \xhookrightarrow{O(\log n)} \ell_p^{O(\log^2 n)} .$$
   Use ideas from the proof for the ℓ_2-embedding. Use Hölders inequality: $\|x\|_p \cdot \|y\|_q \geq \langle x, y\rangle$, for $1/p + 1/q = 1$.

Definition *A metric (X, d) is called a squared ℓ_2-metric if the metric*[1] $d' = \sqrt{d}$ *is an ℓ_2-metric. In the following we denote the set of squared ℓ_2-metrics with $\mathcal{L}_2$.*[2]

3. Show that any ℓ_1-metric embeds into $\mathcal{L}_2$ isometrically.

4. Let H_ℓ denote the hypercube of dimension ℓ, i.e., a graph with vertex set $V(H_\ell) = \{0, 1\}^\ell$ and edge-set $E(H_\ell) = \{(x, y) \in V \times V \mid x \text{ and } y \text{ differ in exactly one bit}\}$. Let d denote the shortest-path metric on H_ℓ. Show that d is an $\mathcal{L}_2$-metric. How many dimension do you need for your embedding?

5. A metric d can be viewed as a function $d : X \times X$ that assigns a length to each unordered pair of points in a space X. We can encode this function by a point in $\mathbb{R}^k$ with $k = \binom{|X|}{2}$, i.e., any metric on n vertices corresponds to a point in $\mathbb{R}^{\binom{n}{2}}$.
   (a) Show that the set of points in $\mathbb{R}^k$ that correspond to a metric forms a convex cone.
   (b) What about points corresponding to ℓ_1-metrics, ℓ_2-metrics, ℓ_∞-metrics, $\mathcal{L}_2$-metrics? For each case either prove that it forms a cone, or find a counter-example.

6. Prove that any n-point metric that is embeddable into ℓ_1 with *any* number of dimensions can actually be embedded into $\ell_1^{\binom{n}{2}}$, i.e., using only $\binom{n}{2}$ dimensions. Use the result for ℓ_1-metrics from the previous question, the fact that ℓ_1-metrics are cut-metrics, and apply some linear algebra.

[1] verify that d' is indeed a metric.
[2] In the literature the set of squared ℓ_2-metrics is often denoted with ℓ_2^2.