# 11 Obtain $\ell_1$-embeddings from cutting schemes

In this lecture we show that the cutting schemes by Bartal and FRT that were used to get an embedding into a distribution over dominating trees can also be used to obtain a subset-embedding with low $\ell_1$-distortion.

**Definition 11.1** *A $\beta$-cutting scheme is a randomized procedure that gets as input a graph $G = (V, E)$ and a paramter $\delta$, and outputs a partitioning $C_1, C_2, \ldots$ of $V$ such that*

- *each component has weak diameter at most $\delta$, and*
- *with constant probability a node is far away from the boundary of its component, i.e.,*

$$\mathbf{Pr}[B(x, \beta \cdot \delta) \text{ is cut}] \leq 1/2 \ ,$$

*where $C(x)$ denotes the component that contains $x$.*

Note that the cutting schemes presented in the previous class fulfill the above definition for $\beta = \Theta(\frac{1}{\log n})$. Next, we will show how to use the cutting scheme to generate good subset embeddings for metrics. In particular, one can prove an approximate version of Bourgain's theorem using the cutting schemes given in class.

We will work with the same assumptions as in the previous lectures, i.e., that all edge lengths are one, and the diameter of the graph $G$ (which represents the metric $(X, d)$) is $\Delta$. We will show how to use a partitioning scheme to prove the following theorem:

**Theorem 11.2** *There exists a subset embedding $\phi_1$ such that*

$$(X, d) \xhookrightarrow{O(\frac{\log \Delta}{\beta})} \ell_1^{O(\log n \log \Delta)} \tag{11.1}$$

Suppose, we wanted to maintain the distance between $u$ and $v$ which are at distance $\delta < d(u, v) \leq 2\delta$, we could do the following: perform a random partitioning with parameter $\delta$ and consider the sum of the distances of $u$ and $v$ from the boundary of the components they lie in. Since the partition ensures that the resulting components have diamater at most $\delta$, the vertices $u$ and $v$ lie in different components in the partition; furthermore, the above observation implies that with constant probability, at least one of $u$ and $v$ lie at distance $\rho = \Omega(\beta \cdot \delta)$ from the boundary of the clusters.

Hence, if we look at the $\log \Delta$ partitions done with $\delta = 2^i$ for $i = 1, 2, \ldots, \log \Delta$, for the particular choice of $\delta \approx d(u, v)$, the sum of the distances of $u$ and $v$ from the boundaries of their clusters is a reasonable approximation ($\approx \beta \cdot d(u, v)$) to their real distance. We now have to figure out how to emulate this by embeddings into $\ell_1$.

We define the mapping $\phi_1$:

1. For each *level* $i = 0, \ldots, \log \Delta$,
2. For $j = 1, \ldots, O(\log n)$,

Use the above scheme to obtain a random partition $\Pi_{ij}$ of the graph $G$ with $\delta_i = 2^i$. For each connected component $C_1, C_2, \ldots, C_{k(i,j)}$ formed in the partition $\Pi_{ij}$, choose a random color $\mathrm{col}(C_l) \in \{0, 1\}$.

For $u \in V$, let $\mathrm{col}(u)$ be the color of the connected component in $\Pi_{ij}$ that contains $u$. Let $V_1$ denote the set of all nodes $x$ with $\mathrm{col}(x) = 1$.

$$f_{i,j}(u) := d(u, S_1.$$

3. Finally, set $\phi_1(u) := \bigoplus_{i,j} f_{i,j}(u)$.

What is the distortion of this embedding? First, let us observe that this subset embedding is a contraction.

**Observation 11.3** *Since, we are dealing with a subset embedding every coordinate mapping $f_{ij}$ is a contraction.*

This claim immediately implies that

$$\|\phi_1(u) - \phi_1(v)\|_1 = \sum_{i,j} |f_{i,j}(u) - f_{i,j}(v)| \leq O(\log n \log \Delta) \cdot d(u,v) \quad . \tag{11.2}$$

Next, we will argue that for all $u$ and $v$, the distances in the embedding is at least a large fraction of $d(u, v)$.

**Claim 11.4** *$|\phi_1(u) - \phi_1(v)|_1 \geq \Omega(\beta \cdot d(u,v) \cdot \log n)$ with probability $1 - n^{-3}$.*

**Proof.** The proof uses Chernoff bounds. Let $d(u, v) \in (2^i, 2^{i+1}]$, and consider the partition $\Pi_{ij}$. Let $X_j$ be the indicator variable for the event

$$\mathcal{E} = \mathsf{Ball}(u, \beta \cdot 2^i) \text{ is not cut by } \Pi_{ij} \quad .$$

Note that the properties of the cutting scheme imply that $\mathbf{Pr}[X_j = 1] \geq 1/2$.

Now if event $\mathcal{E}$ were to happen, then it holds that $|f_{i,j}(u) - f_{i,j}(v)| \geq beta \cdot 2^i$ with probability $1/2$; indeed, with probability $1/2$, the node $u$ is assigned color "1" and with probability $1/2$ the node $v$ is assigned color "0". In this case $|f_{i,j}(u) - f_{i,j}(v)| \geq \beta \cdot 2^i$. Since, all events are indepent we conclude that with probability $1/8$ the coordinate $ij$ gives a contribution of $\beta \cdot 2^i$ to the distance between $u$ and $v$.

Let $Y_j$ denote the indicator variable for the event that the coordinate $ij$ gives contribution $\beta \cdot 2^i$. We can use the Chernoff-Hoeffding bounds and argue that, since each $Y_j = 1$ with constant probability, it holds that for some constant $c > 0$

$$\mathbf{Pr}\left[\sum_{j=1}^{O(\log n)} Y_j \geq c \log n\right] \geq 1 - \frac{1}{n^3} \quad .$$

This immediately implies that

$$\mathbf{Pr}\left[\sum_{j=1}^{O(\log n)} |f_{i,j}(u) - f_{i,j}(v)| \geq \Omega(\beta \cdot d(u,v) \cdot \log n)\right] \geq 1 - \frac{1}{n^3} \ ,$$

and hence $|\phi_1(u) - \phi_1(v)|_1 \geq \Omega(d(u,v))$ with probability $1 - 1/n^3$, thus proving the claim. ■

By taking a union bound we get that with probability at least $1 - 1/n$ all pairs have an $\ell_1$-distance at least $\Theta(\beta \cdot d(u,v) \cdot \log n)$. To get the distortion of the embedding we have to divide the upper bound (Equation 11.2) on the length by the lower bound which gives $\Theta(\frac{\log \Delta}{\beta})$.

We can improve on this result by interpreting $\phi$ as an embedding into $\ell_2$. Then the length is bounded by

$$\|\phi(u) - \phi(v)\|_2^2 \leq \sqrt{\log n \log \Delta} \cdot d(u,v) \ ,$$

since each coordinate is contracting. For the lower bound we again observe that at least a logarithmic number of coordinates have $|f_{ij}(u) - f_{ij}(v)| \geq \Theta(\beta \cdot d(u,v))$, which results in an $\ell_2$ distance of at least $\sqrt{O(\log n)} \cdot \beta \cdot d(u,v)$. Dividing the upper bound by the lower bound gives an embedding with distortion $O(\sqrt{\log \Delta}/\beta) = O(\sqrt{\log n}/\beta)$. Since $\ell_2$ embeds into $\ell_1$ isometrically we also get an $\ell_1$ embedding with this distortion.

In the coming lectures we will first see how to improve this result to a distortion of $O(\sqrt{\log n/\beta})$, then we will show how to construct cutting schemes with a $\beta = \Theta(1)$ for planar graphs, and finally we show how to obtain cutting schemes with $\beta = \frac{1}{\sqrt{\log n}}$ for squared $\ell_2$-metrics.

## References


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