17 Measure Concentration for the Sphere

In today's lecture, we will prove the *measure concentration theorem for the sphere*. Recall that this was one of the vital steps in the analysis of the Arora-Rao-Vazirani approximation algorithm for sparsest cut. Most of the material in today's lecture is adapted from Matousek's book [Mat02, chapter 14] and Keith Ball's lecture notes on convex geometry [Bal97].

Notation: We will use the notation $\mathcal{B}_n$ to denote the ball of unit radius in $\mathbb{R}^n$ and $\mathcal{S}^{n-1}$ to denote the sphere of unit radius in $\mathbb{R}^n$. Let μ denote the normalized measure on the unit sphere (i.e., for any measurable set $S \subseteq \mathcal{S}^{n-1}$, $\mu(A)$ denotes the ratio of the surface area of μ to the entire surface area of the sphere $\mathcal{S}^{n-1}$). Recall that the n-dimensional volume of a ball of radius r in $\mathbb{R}^n$ is given by the formula $\mathrm{Vol}(\mathcal{B}_n) \cdot r^n = v_n \cdot r^n$ where

$$\begin{aligned} v_n &= \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \\ \text{where } \Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}dt \end{aligned}$$

The surface area of the unit sphere $\mathcal{S}^{n-1}$ is nv_n.

Theorem 17.1 (Measure Concentration for the Sphere $\mathcal{S}^{n-1}$) *Let $A \subseteq \mathcal{S}^{n-1}$ be a measurable subset of the unit sphere $\mathcal{S}^{n-1}$ such that $\mu(A) = 1/2$. Let A_δ denote the δ-neighborhood of A in $\mathcal{S}^{n-1}$. i.e., $A_\delta = \{x \in \mathcal{S}^{n-1} | \exists z \in A, ||x-z||_2 \leq \delta\}$. Then,*

$$\mu(A_\delta) \geq 1 - 2e^{-n\delta^2/2}.$$

Thus, the above theorem states that if A is any set of measure 0.5, taking a step of even $O\left(1/\sqrt{n}\right)$ around A covers almost 99% of the entire sphere.

We will give two different (but very related) proofs of this theorem in today's lecture. Both these proofs will use the Brun-Minkowski Theorem, an important tool in convex geometry.

17.1 Brun-Minkowski Theorem

Notation: The Minkowski sum of two sets A and B denoted by $A+B$ is defined as follows:

$$A + B = \{a + b | a \in A, b \in B\}.$$

Theorem 17.2 (Brun-Minkowski Theorem) *For all non-empty measurable subsets A, B of $\mathbb{R}^n$ and any $0 \leq \lambda leq 1$,*

$$\mathrm{Vol}\Bigg((1-\lambda)A + \lambda B\Bigg)^{1/n} \geq (1-\lambda)\mathrm{Vol}(A)^{1/n} + \lambda \mathrm{Vol}(B)^{1/n}.$$

For a proof of this theorem, see Matousek [Mat02, Theorem 12.2.2]

Corollary 17.3 *For all measurable sets A, B of $\mathbb{R}^n$, we have*

$$\mathrm{Vol}\left(\frac{A+B}{2}\right) \geq \sqrt{\mathrm{Vol}(A)\mathrm{Vol}(B)}.$$

Proof.

$$\begin{aligned}\mathrm{Vol}\left(\frac{A+B}{2}\right) &\geq \left[\frac{\mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}}{2}\right]^n \\ &\geq [\mathrm{Vol}(A) \cdot \mathrm{Vol}(B)]^{n/2}\end{aligned}$$

■

Though this corollary is weaker than the Brun-Minkowski theorem, it happens to be a more convenient form to use since it does not involve the dimension n of the ambient space.

As an aside, we will show that the classical isperimetry theorem follows directly from the Brun-Minkowski theorem

Theorem 17.4 (Classical Isoperimetry Theorem) *Among all bodies of the same volume, the ball has the least surface area*

Proof. let $A \subset \mathbb{R}^n$ be any measurable body in $\mathbb{R}^n$ and let $\mathcal{B}$ be a ball in $\mathbb{R}^n$ of the same volume as A. Let r be the radius of the ball $\mathcal{B}$. It can easily be seen that the surface area of A and $\mathcal{B}$ can be expressed as follows:

$$\begin{aligned}\text{Surface-Area}(A) &= \lim_{\epsilon \to 0} \frac{\mathrm{Vol}(A + \epsilon\mathcal{B}_n) - \mathrm{Vol}(A)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\mathrm{Vol}(A_\epsilon) - \mathrm{Vol}(A)}{\epsilon} \\ \text{Surface-Area}(\mathcal{B}) &= \lim_{\epsilon \to 0} \frac{\mathrm{Vol}(\mathcal{B} + \epsilon\mathcal{B}_n) - \mathrm{Vol}(\mathcal{B})}{\epsilon} = \lim_{\epsilon \to 0} \frac{\mathrm{Vol}(\mathcal{B}_\epsilon) - \mathrm{Vol}(\mathcal{B})}{\epsilon}\end{aligned}$$

It suffices for us to show that $\mathrm{Vol}(A_\epsilon) \geq \mathrm{Vol}(\mathcal{B}_\epsilon)$. By, the Brun-Minkowski Theorem

$$\begin{aligned}\mathrm{Vol}(A_\epsilon) &= \left[\mathrm{Vol}(A + \epsilon\mathcal{B}_n)^{1/n}\right]^n \\ &\geq \left[\mathrm{Vol}(A)^{1/n} + \epsilon\mathrm{Vol}(\mathcal{B}_n)^{1/n}\right]^n \\ &= \left[\mathrm{Vol}(\mathcal{B})^{1/n} + \epsilon\mathrm{Vol}(\mathcal{B}_n)^{1/n}\right]^n \\ &= \left[v_n^{1/n} \cdot r + \epsilon v_n^{1/n}\right]^n \\ &= v_n \cdot (r + \epsilon)^n \\ &= \mathrm{Vol}(\mathcal{B}_\epsilon)\end{aligned}$$

■

An isometry theorem for the sphere (instead of $\mathbb{R}^n$), which looks deceivingly similar to the above, but far more difficult to prove, was shown by P. L'evy.

Theorem 17.5 (Isoperimetry Theorem for Sphere) *Let $A \subset \mathcal{S}^{n-1}$ be a measurable subset of the unit sphere $\mathcal{S}^{n-1}$ and let C be a spherical cap on the sphere $\mathcal{S}^{n-1}$ of the same surface area as A, then for all $\epsilon > 0$,*

$$\mu(A_\epsilon) \geq \mu(C_\epsilon).$$

Or equivalently, among all (measurable) patches on the unit sphere, the spherical cap has the least perimeter.

17.2 First Proof oF Theorem 17.1

Let us first consider the easy case when A is the hemisphere. In this case, we need to show that the spherical cap that is at distance δ from the hemisphere (i.e., the complementary cap to A_δ, see Figure 17.1(a)) has normalized measure at most $2e^{-n\delta^2/2}$.

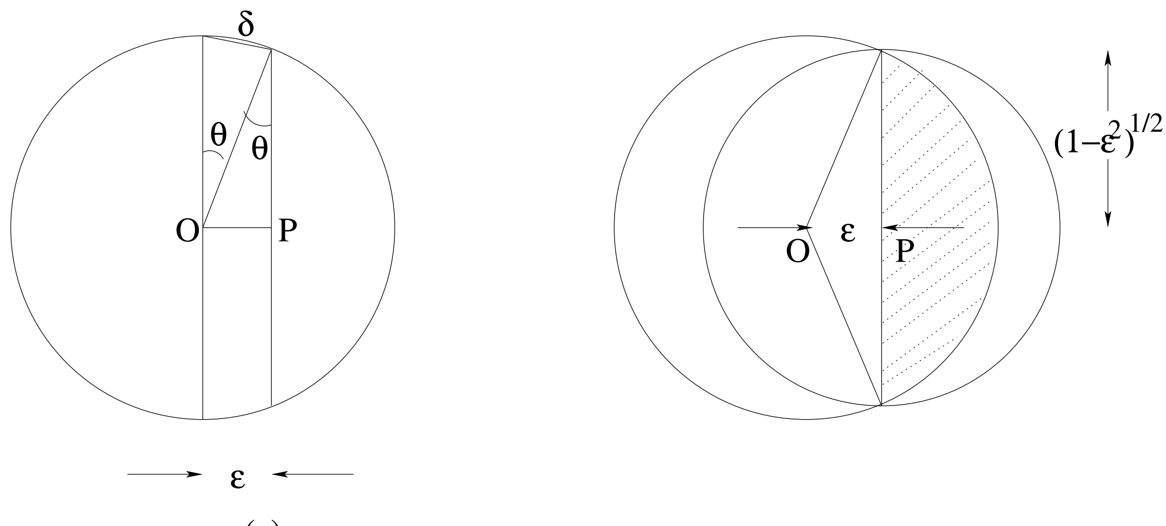

Figure 17.1: Estimating area of a spherical cap

Let ϵ be the distance from the center of the sphere to the plane that cuts the sphere into the cap A_δ and its complementary cap (see Figure 17.1(a)). We observe that $\epsilon = \sin\theta$ and $\delta = 2\sin(\theta/2)$. Hence, $\epsilon = \delta \cdot \cos\theta/2 \geq \sqrt{2}\delta$.

The normalised measure of the cap (ie., $1 - \mu(A_\delta)$) is exactly the ratio of the volume of the cone C subtended by the cap A_δ at the center O of the sphere to the volume of the sphere. Consider another $\mathcal{S}'$ centered at P with radius $\sqrt{1-\epsilon^2}$ (see Figure 17.1(b)). The cone C is clearly completely contained in this sphere. Hence, $\mu(A_\delta)$ is at most the ratio of the volume of the sphere of radius $\sqrt{1-\epsilon^2}$ and the volume of the unit sphere $\mathcal{S}^{n-1}$ which is at most $(1-\epsilon^2)^{n/2} \leq e^{-n\epsilon^2/2} \leq e^{-n\delta^2}$. This proves the case when A is a shperical cap.

Now to the more general case, when A is not necesarily a spherical cap. However, the Isoperimetry Theorem for the Sphere 17.5 states that the measure of A_δ is even larger than that of C_δ where C is a spherical cap of the same measure as A. Hence, $\mu(A_\delta) \geq \mu(C_\delta) \geq 1 - e^{-n\delta^2}$. This completes the first proof of the Measure Concentration of the Sphere.

17.3 Second Proof of Theorem 17.1

Our second proof of the Measure Concentration will directly follow from the Brun-Minkowski Theorem.

Let A be any measurable surface on the unit sphere with $\mu(A) = 1/2$. Let B be the surface on the unit sphere $\mathcal{S}^{n-1}$ containing points at least δ-far from A (in other words, $B = \mathcal{S}^{n-1} \setminus A_\delta$). Consider the following sets $\tilde{A}, \tilde{B} \subset \mathcal{B}_n$ defined as follows:

$$\begin{aligned}\tilde{A} &= \{\alpha x | x \in A, \alpha \in [0,1]\} \\ \tilde{B} &= \{\alpha x | x \in B, \alpha \in [0,1]\}\end{aligned}$$

Observe that $\mathrm{Vol}(\tilde{A}) = \mu(A) \cdot v_n = v_n/2$ and similarly $\mathrm{Vol}(\tilde{B}) = \mu(B) \cdot v_n$.

We will bound the distance of any point $(\tilde{a} + \tilde{b})/2$ from the origin for any $\tilde{a} \in \tilde{A}$ and $\tilde{b} \in \tilde{B}$. This distance is maximized when $\tilde{a} \in A$ and $\tilde{b} \in B$ and furthermore, when a and b are as close to each other as possible. However, by defintion, for any point $a \in A$ and $b \in B$, we have $||a - b||_2 \geq \delta$. This implies that for all points $\tilde{a} \in \tilde{A}$ and $\tilde{b} \in \tilde{B}$, we have $||(\tilde{a} + \tilde{b})/2||_2 \leq \sqrt{1 - \delta^2/4} \leq 1 - \delta^2/8$. This implies that all points in the body $(\tilde{A} + \tilde{B})/2$ is contained in the sphere of radius $1 - \delta^2/8$. By Corollary 17.3 to the Brun-Minkowski Theorem, we have

$$\begin{aligned}\sqrt{\mathrm{Vol}(\tilde{A}) \cdot \mathrm{Vol}(\tilde{B})} &\leq \mathrm{Vol}\left(\frac{\tilde{A} + \tilde{B}}{2}\right) \\ \Longrightarrow \sqrt{\frac{v_n}{2} \cdot \mu(B) v_n} &\leq \left(1 - \frac{\delta^2}{8}\right)^n \cdot v_n \\ \Longrightarrow \mu(B) &\leq 2\left(1 - \frac{\delta^2}{8}\right)^{2n} \\ &\leq 2e^{-n\delta^2/4}\end{aligned}$$

This proves that $\mu(A_\delta) \geq 1 - 2e^{-n\delta^2/4}$ which is slightly weaker than what we set out to prove.

References

[Bal97] Keith Ball. An elementary introduction to modern convex geometry. Technical Report 31, MSRI Publications, 1997.

[Mat02] Jiří Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer, New York, 2002.